Trait-Augmented Games with Limited-Skill Agents

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Abstract

We present a bounded-rationality framework in which agents are unable to distinguish the available actions by direct observation. Instead they view a limited number of traits and use Bayes' rule in an attempt to distinguish the actions. This framework can be fruitfully grafted into many applications. Three examples are presented: the relationship between skill and expected payoff, matching, and product differentiation.

Keywords: bounded rationality, skill, matching, product differentiation.

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1 Introduction

Experiments and empirical studies based on notions of bounded rationality—a term attributed to Herbert Simon (1957)—have called aspects of economic orthodoxy into question.[1] A deep understanding of these anomalies is likely to be grounded in theoretical frameworks that explicitly model the underlying causes of departures from conventional rationality.[2]

The theoretical literature that departs from classical rationality has several strands. Evolutionary game theory requires no rationality at all and argues by analogy to the biological sciences that successful behavior will tend to persist.[3] Learning models focus on individuals who update their beliefs as they gain experience and acquire information.[4] Other strands study the implications of various decision rules.[5]

The studies most in the spirit of our own explicitly model a limit on some aspect of players' abilities. Players are modeled as taking into account

[1] See Camerer, Loewenstein, and Rabin (2004) for a review of some of the earlier empirical literature. Some recent papers include, among many others, Goldin (2015), Chetty, Looney, and Kroft (2009), Apicella et. al. (2014), Sip et. al. (2015), and Augenblick (2016).

[2] See Harstad and Selten (2013).

[3] See, for example, Samuelson (1998).

[4] See, for example, Roth and Erev (1995), Jéheil and Samet (2005), Fudenberg and Levine (1993), and Lambson and Probst (2004).

[5] Chen, Su, and Zhao (2012), McKelvey and Palfrey (1995), and McKelvey and Palfrey (1998) employ quantal response equilibria (wherein players are more likely to pick a better response but are unable to recognize and implement the optimal response with certainty).

their own limitations as well as those of the other players.[6] Specifically, players are aware of their available actions but cannot identify them. Thus the emphasis is on the difficulty of deducing the best action.[7] It is as if the moves are labeled—and hence distinguishable—but the labels are then randomly permuted. The extent of bounded rationality is defined by the ability to observe and interpret *traits*, which each action exhibits and which players may view in an effort to identify the correct labeling. We define *skill* in terms of the number of traits which an agent is able to observe and interpret. Players who observe fewer traits are less skilled. Players who are thus limited are called limited-skill players or players of limited skill. Limited-skill players can be grafted into many contexts. Section 3 provides some examples:

First, we consider the relationship between skill and payoffs. Conditions under which more skilled players achieve greater expected payoffs are stringent. Even strong coordination games and games with dominant strategies do not generally exhibit this property.[8] Two-player constant sum games and

[6]Gabaix (2014) imposes a cost of attention, with agents maximizing subject to limited attention to parameters. Monte (2014) studies bounded memory in repeated games with one-sided incomplete information. Saran (2016) analyzes bounded depths of rationality - an understanding of how rational other players are. Jéheil (1995) studies infinite-horizon alternating-moves games where players can see only finitely many moves ahead.

[7]This is in contrast to Abreu and Rubinstein (1988) who focus on the difficulty of implementing complicated strategies.

[8]We use *strong coordination* games to refer to games wherein players agree on the ranking of *all* outcomes and not just equilibrium outcomes.

reciprocal games exhibit a relationship between skill and payoffs.[9]

Next we consider matching models where players may have information about only some of the traits of some of the potential partners. We establish conditions under which mismatches occur with strictly positive probability, and explore the relationship between skill and probability of mismatching. It is also shown by example that both sides may have incentives to misrepresent true preferences. This is in contrast to the classical deferred acceptance paradigm in which only one side may have incentives to misrepresent preferences.

Finally, we consider a model of product differentiation when consumers are of limited skill. We show that if and only if marginal costs are sufficiently low, product differentiation is greater with limited-skill consumers because it contributes to consumers' ability to distinguish available products. We also show that, whether consumers are of limited-skill or fully rational, product differentiation is greater than socially optimal.

2 Cognitive limitations

An agent is to choose an *action*, a, from a set A. The agent knows the payoff function, $V : A \to \mathbb{R}$, but is unable to distinguish the actions. It is as if the actions were re-indexed by $m \in M = A$ and that the agent is unaware of the re-indexing rule that was applied. The re-indexed actions will be called

[9] We use *reciprocal games* to refer to two-player games wherein a change in one player's strategy changes both players' payoffs in the same direction regardless of the mixed strategy played by the other player. These results for two-player constant-sum games generalize Lambson and van den Berghe (2015, Theorem 3.1).

moves.

A *trait*, $\tau \in T$, is a list of probability distributions $\{\phi(\cdot|a)\}_{a \in A}$. The agent draws a list of move-trait pairs, $\eta = \{(m^1, \tau^1), ..., (m^k, \tau^k)\}$, called a *consideration set*. The cardinality of the chosen consideration set cannot exceed an exogenous, non-negative integer, k, that is interpreted as a measure of the agent's *skill*: more skilled agents are able to consider more move-trait pairs. Moves are equally likely to be chosen due to the agent's inability to distinguish them *ex ante*.

An *observation set* $\{\theta^1, ..., \theta^k; \eta\} := \theta \in \Theta$ is generated by a consideration set, η, where $\{\theta^1, ..., \theta^k\}$ is a list of *observations* independently drawn from each probability distribution comprising η. Given θ, Baye's formula defines a probability distribution over A for each m. Let $\mu(a|\theta, m)$ denote the probability that the move, m, is the action, a, given the observation set, θ. The agent then chooses m. The expected value of choosing m given θ is

$$\Gamma(m|\theta) = \sum_{a \in A} V(a)\mu(a|\theta, m)$$

Given the realization of θ, the expected-payoff maximizing agent chooses m to maximize $\Gamma(m|\theta)$. Anticipating this, the expected payoff maximizer chooses η to maximize

$$\sum_{\theta \in \Theta} \phi(\theta|\eta) \max_m \Gamma(m|\theta)$$

In summary, an agent chooses a consideration set comprised of move-trait pairs, observes the realization of the observation set randomly generated by the consideration set, applies Bayes' formula to generate a probability distribution over the action set A for each m, and chooses an move m. This framework can be applied to any classical game theoretic model: extensive-form games, simultaneous move games, etc. Agents as described above will be called *limited-skill* players. A game for which actions are endowed with traits as described above will be called *trait-augmented* games.

3 Applications

We find it plausible that the behavior of limited-skill individuals in trait-augmented captures aspects of real-world decision making. Grafting these individuals into classical models generates insights. This section contains some examples. They are offered by way of illustration. No attempt is made to achieve the greatest possible generality.

3.1 Skill and Payoffs

Consider a trait-augmented, finite, extensive form game. Let $N = \{1, ..., n\}$ be the set of n (limited-skill) players, let H_t be the set of $t-$period histories, $H = \cup_{t=1}^{T} H_t$, ι_h be the active player at the node h, let A_{ι_h} be the active player's action set at the node $h \in H$, let Z be the set of end nodes, and let $\pi_i : Z \to \mathbb{R}$ be player i's payoff function.

As in section 2, the active player is informed which node is current, chooses a consideration set, sees the realized observation set, and chooses a move. A strategy for player i is a list of pairs $\sigma_i = \{\eta_i(h), m_i(\theta, h)\}$ for all h where i is active. We restrict attention to pure strategies for the choice of η. We assign a list of values $V(h) \in R^n$ to each node $h \in H$ by backward induction as follows.[10]

We assign $(\pi_1(z), ..., \pi_n(z))$ to each end node z. Then, having assigned values to all n for each $g \in H_{t+1}$, the active player ι_h at $h \in H_t$ is in a situation as described in section 2. We assume the active player chooses randomly with equal probability among the elements of $\arg_m \max \Gamma(m|\theta)$ if the set is not a singleton. The assigned values $(V_1(h), ..., V_n(h))$ are:

$$V_j(h) = \sum_{\theta \in \Theta} \phi(\theta|\eta) \sum_{m \in \text{argmax } \Gamma(m|\theta)} \frac{\sum_{a \in A} V_j(a)\mu(a|\theta, m)}{\# \arg_m \max \Gamma(m|\theta)}$$

An equilibrium is a list of strategies such that for all j, and for all h at which j is active, $m_j(\theta, h)$ maximizes $\Gamma(m|\theta)$, and $\eta_j(h)$ maximizes $\sum_{\theta \in \Theta} \phi(\theta|\eta) \max_m \Gamma(m|\theta)$.

It is well-known that the ability to commit to irrational strategies can increase payoffs. Therefore it is not surprising that there is no general relationship between skill and payoffs because increases in skill tend to move agents toward full rationality. Conditions under which a player benefits from an increase in own skill are stringent.

Some insight can be gained by decomposing the effects of skill increases on payoffs into two channels: a decision theoretic channel and a game-theoretic channel. Let $\sigma^{k_i} = (\sigma_1^{k_i}, ..., \sigma_n^{k_i})$ denote equilibrium strategies when i has skill k_i and let $W_i(\sigma_i, \sigma_{-i})$ be the expected payoff to i. Specifically, $W_i(\sigma_i, \sigma_{-i}) = \sum_z Pr(z|\sigma)V_i(z)$ where $Pr(z \mid \sigma)$ is the probability of reaching terminal node z when strategies σ are employed. The effects of skill on payoff are decomposed

[10] The use of $V(h)$ and $V(a)$ entails a slight abuse of notation because a is an action and h is a history of actions (a node). So $V(h)$ is the value of arriving at the node h, $V(a)$ is the value of choosing the node associated with the action a. However, if the action a results in the node h, $V(h) = V(a)$.

as:

$$\Delta\pi_i = \left[W_i(\sigma_i^{k_i}, \sigma_{-i}^{k_i+1}) - W_i(\sigma_i^{k_i}, \sigma_{-i}^{k_i})\right] + \left[W_i(\sigma_i^{k_i+1}, \sigma_{-i}^{k_i+1}) - W_i(\sigma_i^{k_i}, \sigma_{-i}^{k_i+1})\right]$$

The first term represents the game theoretic channel, that is, the effect on player i's payoff caused by agents other than player i. The second term represents the decision theoretic effect of skill, that is, the effect caused by player i's switch to the new equilibrium given the new equilibrium strategies of the other players and the additional skill. The first term is of ambiguous sign; the second term is always (weakly) positive. Thus an increase in skill increases an agent's expected payoff if the first term is not too negative.

3.1.1 Strong Coordination Games

We define strong coordination games as games in which for all i, j, z, z',

$$\pi_i(z) > \pi_i(z') \iff \pi_j(z) > \pi_j(z').$$

It may seem plausible that with completely aligned interests, an increase in an agents' skill would benefit all players. That it need not benefit even the recipient of the skill increase is established by the following counterexample:

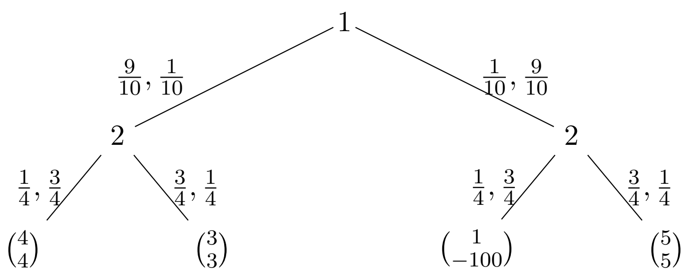

Figure 1: A strong coordination game. There is a single trait at each node with support $\{A, B\}$. The first fraction on each node represents the probability A is realized, the second represents the probability B is realized.

Suppose player 1 has $k_1 = 1$, and player 2 has $k_2 = 0$.[11] Then player 1 optimizes by moving left with the highest possible probability, which is 90%. The expected payoff to player 1 is then $.9*(4*.5+3*.5)+.1*(1*.5+5*.5) = 3.45$ and to player 2 is $.9*(4*.5+3*.5)+.1*(-100*.5+5*.5) = -1.6$.

Suppose instead that $k_2 = 1$. At each node where player 2 is active, player 2 can select the more desirable action with probability $\frac{3}{4}$. Now player 1 optimizes by maximizing the probability of moving right to obtain $.1*(4*.75+3*.25)+.9*(1*.25+5*.75) = 3.975$. Player 2 in expectation will attain a payoff of $.1*(4*.75+3*.25)+.9*(-100*.25+5*.75) = -18.75$. Thus the increase in player 2's skill generates a decrease in player 2's payoff because the negative game theoretic channel outweighs the positive decision theoretic channel:

$$\Delta\pi_2 = \left[W_2(\sigma_2^0, \sigma_1^1) - W_2(\sigma_2^0, \sigma_1^0)\right] + \left[W_2(\sigma_2^1, \sigma_1^1) - W_2(\sigma_2^0, \sigma_1^1)\right]$$

[11] That is, player 2 is a random mover.

$$= \left[-42.4 - (-1.6) \right] + \left[-18.75 - (-42.4) \right] = \left[-40.8 \right] + \left[23.65 \right] = -17.15$$

3.1.2 Dominant Strategies

It may seem plausible that if a player's opponent has a dominant strategy—and would not wish to change strategies in the full rationality game—then a player's payoff should be positively related to skill. Consider the following counterexample:

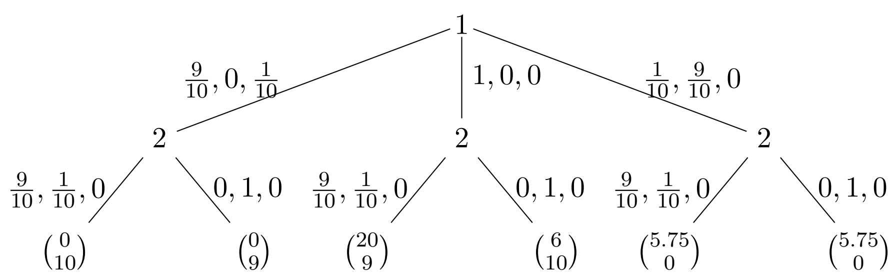

Figure 2: There is a single trait at each node with support {A, B, C}, the elements of the triples represent the respective probabilities.

Player 1 has a dominant strategy in the full rationality game, namely, middle. Suppose player 1 has $k_1 = 3$ and player 2 has $k_2 = 0$. The equilibrium exhibits player 1 including each move-trait pair in η, and choosing randomly among all moves from which A is observed (which may be a singleton set). Player 2 is a random mover. Expected payoffs are 7.17625 and 9.2325.

Should player 2 instead have $k_2 = 1$, player 1 optimizes by choosing the move from which B is observed, or randomly among moves from which A is observed if B is not observed. Player 2, when active, is able to choose the more desirable action with probability 95%. This results in expected payoffs to player 1 and player 2 respectively of 5.61075 and .64675.

The change in player 2's payoff is negative and can be decomposed into the game theoretic and decision theoretic channels:

$$\Delta\pi_2 = \Big[W_2(\sigma_2^0, \sigma_1^1) - W_2(\sigma_2^0, \sigma_1^0)\Big] + \Big[W_2(\sigma_2^1, \sigma_1^1) - W_2(\sigma_2^0, \sigma_1^1)\Big]$$

$$= \Big[.6175 - 9.2325\Big] + \Big[.64675 - .6175\Big] = \Big[-8.615\Big] + \Big[.02925\Big] = -8.58575$$

The game theoretic channel dominates and is negative because player 1 is rarely able to identify the dominant strategy, so the incentive to play the dominant strategy is less relevant.

3.1.3 Two-Player Constant Sum Games

The previous two counterexamples suggest that general results are unattainable. However, there are non-trivial classes of games that exhibit a positive relationship between skill and payoffs. For example, there is proposition 3.1.

Proposition 3.1 In a two-player constant sum, extensive form, trait-augmented game with limited-skill agents, player i's equilibrium expected payoff is non-decreasing in k_i.

Proof: Consider two equilibria in which player i has skill of k_i and $k_i + 1$, respectively, other things the same. Let z^{-1} be a penultimate node. There are two cases. (1) If player i is active at z^{-1} then additional skill cannot decrease player i's expected payoff because player i needn't choose a larger consideration set. Therefore the expected payoff to player i is weakly greater in the equilibrium with skill $k_i + 1$. (2) If $j \neq i$ is active then player i's expected payoff at z^{-1} is independent of player i's skill. Therefore expected payoffs upon reaching z^{-1} are the same in the two equilibria. In either case, the expected payoff to player i of reaching node z^{-1} is weakly greater in the equilibrium with skill $k_i + 1$.

Assume that player i's equilibrium expected payoffs are weakly higher when player i's skill is $k_i + 1$ for the subgames beginning at all $h \in H_{t+1}$. It must be shown for the subgames beginning at all $h \in H_t$ that expected payoff to i is weakly greater in the equilibrium for which i has skill $k_i + 1$.

There are two cases. (1) If i is active at $h \in H_t$, note again that i needn't choose a larger consideration set. Because expected payoff of each possible action is weakly greater, expected payoff to i is weakly greater in the equilibrium with skill $k_i + 1$. (2) If player $j \neq i$ is active at $h \in H_t$ then, note that because the game is constant sum, the expected payoff of all possible actions to j is weakly lower in the equilibrium when i has skill of $k_i + 1$. Therefore the expected payoff to j of reaching a node $h \in H_t$ is weakly lower in the equilibrium with skill $k_i + 1$. Because the game is constant sum, the expected payoff to i of reaching the node is weakly greater in the equilibrium with skill $k_i + 1$.

It follows by induction that player 1's expected payoff is weakly greater in the equilibrium for which i has skill $k_i + 1$. QED

3.1.4 Reciprocal Games

Another class of games for which the result holds is the class we call *reciprocal games*. These are two-player, finite, extensive-form games for which

$$[U_i(\sigma_i, \sigma_j) - U_i(\sigma_i, \sigma_j')][U_j(\sigma_i, \sigma_j) - U_j(\sigma_i, \sigma_j')] \geq 0 \ \forall \ \sigma_j \ \sigma_i \ \sigma_j'$$

where i is the agent exhibiting the skill increase, $j \neq i$, σ_ℓ denotes a mixed strategy for player ℓ in the full-rationality game, and $U_i(\sigma_i, \sigma_j)$ is player i's payoff.

Proposition 3.2 In reciprocal trait-augmented games with limited-skill agents, player i's expected payoff is nondecreasing in k_i.

Proof: Consider equilibria in which player i has skill of k_i and $k_i + 1$, respectively. Let z^{-1} be a penultimate node. By the same reasoning as in proposition 3.1, player i's expected payoff is no less in the subgame beginning at z^{-1} in the equilibrium with skill of $k_i + 1$ compared to the equilibrium with skill of k_i.

Assume that player i's equilibrium expected payoffs are weakly higher when player i's skill is $k_i + 1$ for the subgames beginning at all $h \in H_{t+1}$. It must be shown for the subgames beginning at all $h \in H_t$ that expected payoff to i is weakly greater in the equilibrium for which i has skill $k_i + 1$.

There are two cases. (1) If i is active at $h \in H_t$, note again that i needn't choose a larger consideration set. Because expected payoff of each possible action is greater, expected payoff to i is weakly greater in the equilibrium with skill $k_i + 1$. (2) If player j is active at $h \in H_t$, player j may change strategies depending on i's skill. If player j uses the same strategy when player i has skill of $k_i + 1$ as when player i has skill of k_i, then player i's payoff is weakly greater in the subgame beginning at $h \in H_t$ because all actions exhibit weakly greater expected payoff to i by the induction hypothesis. If player j changes equilibrium strategy when player i has skill of $k_i + 1$ compared to the equilibrium strategy when player i has skill of k_i, it must increase the expected payoff to player j. By being a reciprocal game, this strategy change will also increase the expected payoff to player i.

The result now follows by induction. QED

3.2 Matching

Matching models have been applied to study situations including the matching of workers with employers, the matching of interns with hospitals, the matching of students with schools, and the matching of partners in the dating market.[12] Despite many advancements in this literature sparked by Gale and Shapley (1962), there remains minimal theoretical analysis explaining the frequent occurrence of mismatches in such markets.[13] In this section we graft limited-skill agents into the classical matching model.

Let B and G be sets of agents. We assume $\#B = \#G$ and all agents prefer being matched to being unmatched. A partnership is an element (b, g) of $B \times G$. A *matching* is a list of partnerships, $P \subset B \times G$ such that if $(b, g) \in P$ and $(b', g') \in P$ then $b \neq b'$ and $g \neq g'$. Each g and b has a value defined for each possible partnership, respectively denoted $V_g(g, b) \in \mathbb{R}$ for all $b \in B$ and $V_b(g, b) \in \mathbb{R}$ for all $g \in G$. For a matching, P, if there exists a $b \in B$ and $g \in G$ which both strictly prefer matching with each other rather than to their match in P, each is *mismatched*. A matching is *stable* if there are no mismatched agents.

The classic matching paradigm imposes players' preferences exogenously. Given preferences, the existence of stable matchings for fully rational agents is well known. Which stable matching is reached depends on the mechanism employed. We focus attention on the deferred acceptance algorithm. Specif-

[12] Roth and Sotomayor (1990) provide an excellent introduction to the literature. Recent research in matching includes Fox (2018), Azevedo and Leshno (2016), and Abdulkadiroglu, Agarwal, and Pathak (2017).

[13] One strand of literature focuses on mismatches in the labor market, for example Kagel and Roth (2000) who consider agents matching inefficiently early due to congestion.

ically, each agent $g \in G$ proposes a partnership to his most preferred agent $b \in B$. Each agent b "holds" her most preferred proposal and rejects the others. The next steps occur iteratively. Any agent g rejected at the previous step makes a new proposal to the most preferred agent b who hasn't yet rejected him. Each agent b holds her most preferred offer to date, and rejects the rest. These steps are repeated until all $g \in G$ are held. Each agent g is then matched to the b which holds him. The deferred acceptance algorithm converges to a stable matching. With the assumptions that $\#G = \#B$ and that all agents prefer being matched to unmatched, all agents will match.

We graft limited-skill agents into this framework and adjust the mechanism in the following manner. In each round, all agents $g \in G$ choose a consideration set of b-trait pairs and view the realized observation set. Each agent g can propose to any $b \in B$. All agents $b \in B$ are aware of the number of proposals they receive, however are unable to distinguish *ex ante* which $g \in G$ have proposed. Each agent $b \in B$ creates a consideration set of g-trait pairs from the $g \in G$ which have proposed. They then view the observation set and choose which proposal to hold. The next steps occur iteratively. Each agent $g \in G$ rejected in the previous round creates a consideration set of b-trait pairs from the agents $b \in B$ which have not yet rejected him and view the realized observation set. Observed b-trait pairs from previous rounds are forgotten. Each $g \in G$ then proposes to a $b \in B$ which has not yet rejected him. All $b \in B$ remember only the g-trait pairs for the $g \in G$ they are currently holding, if any. Each $b \in B$ creates a new consideration set of g-trait pairs for $g \in G$ which have proposed that round. Each b observes the new observation set and remembers the observations of the g held from the previous round. Each $b \in B$ then chooses a $g \in B$ which has proposed to hold. The process continues until all $g \in G$ are held. If agents are indifferent it is assumed they choose randomly with equal probability.

The deferred acceptance algorithm imposes that agents propose to, and accept, their most preferred available matches in each round. The analogous strategy with limited-skill agents is that agents maximize the expected utility in each round of the partnership which is proposed or accepted.

We refer to this mechanism with limited-skill agents as the *trait-augmented deferred acceptance* algorithm. We establish conditions under which the matchings that result with limited-skill agents in a trait-augmented deferred acceptance algorithm are unstable with positive probability. These conditions rule out trivial cases for which individuals cannot be mismatched, or for which all agents could act as if they are fully rational.

Proposition 3.3 Assume that for all agents, the support of each trait, τ, is the same. Further assume there exist $\{g_1, g_2, b_1, b_2\}$ such that $V_{g_1}(g_1, b_1) > V_{g_1}(g_1, b_2)$ and $V_{b_1}(g_1, b_1) > V_{b_1}(g_2, b_1)$. Then the probability of at least one mismatch resulting from the trait-augmented deferred acceptance algorithm is positive.

Proof: See Appendix.

Intuitively, consider $\{g_1, g_2, b_1, b_2\}$ such that $V_{g_1}(g_1, b_1) > V_{g_1}(g_1, b_2)$ and $V_{b_1}(g_1, b_1) > V_{b_1}(g_2, b_1)$. If $(g_2, b_1) \in P$ and $(g_1, b_2) \in P$, then g_1 and b_1 are mismatched. There is a positive probability that g_1 and g_2 realize observation sets that induce g_1 to propose to b_2 and g_2 to propose to b_1. Similarly there is a positive probability that b_1 and b_2 realize observation sets that induce b_1 and b_2 to accept proposals from g_2 and g_1 respectively. To complete the proof, it is shown there is a positive probability that b_1 and b_2 will receive no other proposals.

Agents make decision-theoretic choices to propose, or accept a proposal. In this setting it may be hypothesized that increased skill causes an agent to be less likely to be mismatched. Such a result fails because agents do not minimize the probability of being mismatched, but rather maximize expected utility. This is illustrated in the following example.

Suppose $\#G = \#B = 3$. Let the values of g and b for each partnership be as shown in the following table.

Table 1: Probability of Mismatch Increases with Skill.

	b_1	b_2	b_3
g_1	0,1	0,0	1,0
g_2	1,0	0,0	0,1
g_3	0,0	1,1	-100,0

The left number is $V_g(g, b)$, the right number is $V_b(g, b)$.

If g_3 and b_2 are a partnership, there can be no mismatched individuals. However if g_3 and b_2 are not a partnership, both g_3 and b_2 will be mismatched. Therefore the probability g_3 is mismatched is the probability g_3 and b_2 are not a partnership.

Suppose there is a single trait, τ, with support $\{x, y\}$ for which $\phi(\cdot|g)$ is identical for g_1, g_2, and g_3. That is, regardless of skill, no $b \in B$ can distinguish any $g \in G$ and therefore, if receiving multiple proposals, accepts randomly.

Suppose that $\phi(x|b_2) = \phi(x|b_3) = 1$, and $\phi(x|b_1) = 0$. Let g_1, g_2, and g_3 have skill of 0. In this setting, all possible matches are equally likely, and g_3 will be mismatched with probability 2/3.

Now suppose g_3 has skill of 1, all else equal. Because g_3 has such low utility of matching with b_3, and because b_3 and b_2 are indistinguishable, g_3 maximizes expected utility by attempting to match with b_1. So g_3 creates a consideration set of a single move-trait pair.[14] If y is observed, g_3 chooses the move for which the move-trait pair is in the consideration set, namely proposal to b_1, and takes a random draw from other moves if x is observed. It follows g_3 will propose to b_1 in the first round with probability 2/3, and will propose to b_2 and b_3 with probability 1/6 each in the first round. It is straightforward to show that in this setting the probability g_3 matches with b_2 is less than 1/3.

In summary, with skill of 0 the probability g_3 is in a partnership with b_2 is 1/3, and with skill of 1 the probability is less than 1/3. Thus the probability g_3 is mismatched is greater when g_3 has skill of 1.

The final result of this section regards the incentives to accurately represent preferences. It has long been known that in deferred acceptance the agents that propose have a weakly dominant strategy of truthfully reporting preferences. The agents which receive proposals, however, may have incentives to misreport preferences. The analogous possibility in trait-augmented deferred acceptance is that agents might not maximize expected payoffs by maximizing the expected utility in each round of the partnership which is proposed or accepted. Because deferred acceptance is a special case of trait-

[14]In this setting "move" denotes a proposal, or an acceptance, of a potential partnership.

augmented deferred acceptance, it is immediate that there exist conditions under which some $b \in B$ has an incentive to deviate from truth-telling.[15] Interestingly, when agents are of limited skill, the proposing side also can have incentives to misrepresent preferences.

Intuitively if $b \in B$ receives a large number of proposals, she may have minimal ability to distinguish the offers. Suppose g^* and b^* mutually maximize utility by matching together. It may be that g^* optimizes by initially proposing to a less desired $b \in B$ if many other $g \in G$ are proposing to b^*, and b^* has minimal ability to distinguish g^*. By delaying a proposal to b^*, g^* may be able to propose when there are fewer proposals, allowing b^* to identify him and increase the probability of matching.

An example of this follows. Assume there exists a trait for which the support uniquely identifies each agent. The utilities of potential partnerships are presented in table 2.

[15] Deferred acceptance is the special case where there is a trait for which the support uniquely identifies each agent and all $b \in B$ have skill greater than or equal to $\#G$, and all $g \in G$ have skill greater than or equal to $\#B$.

Table 2: Incentive to Misreport

	b_1	b_2	b_3	$99 \times b_4$
$50 \times g_1$	2,2	0,0	0,0	1,1
$50 \times g_2$	0,0	2,2	0,0	1,1
g_3	0,0	0,0	1,1	0,0
g_4	3,3	0,0	0,0	0,0

There are 50 agents of type g_1, 50 of type g_2, and 99 of type b_4. The left number is $V_g(g, b)$, the right number is $V_b(g, b)$.

Suppose all $g \in G$ have skill of 102 and are thus able to identify all $b \in B$, and suppose that all $b \in B$ have skill of 1. If all agents act as described in the trait-augmented deferred acceptance algorithm, g_4 will be one of 51 agents which propose to b_1 in the first round. There is a 2/51 chance that b_1 would accept g_4's proposal. If instead g_4 deviated and proposed to b_3 initially, b_3 will accept the proposal from g_3, who is also proposing. In the second round, all unmatched g_1 and g_2 will propose to a b_4, and g_3 is held by b_3. Therefore in the second round of proposals, g_4 would be the only agent to propose to b_1, and b_1 would accept this proposal.

Thus with bounded rationality there are examples in which proposers have incentives to misreport preferences due to the cognitive limitations on the accepting side.

3.3 Product Differentiation

Heretofore we have considered applications wherein traits serve only to help limited-skill players distinguish actions. The framework can be made richer by allowing traits to affect utility directly and endogenizing the feasible consideration sets. This subsection contains a two-firm example pertaining to heterogeneity in product characteristics.[16]

Product differentiation serves two purposes here: it directly affects the utility of consumers, and it provides a mechanism to distinguish goods. We establish conditions for which the magnitude of product differentiation is greater with limited-skill consumers than with fully rational consumers. We also show that the equilibrium level of differentiation is excessive relative to the socially optimal.

Consider two firms indexed by $f \in \{1, 2\}$, and selling products a^1 and a^2, respectively. There is a unit measure of consumers, indexed by ζ, having skill $k = 1$ and heterogeneous utility functions. Consumers observe a single product-trait pair and purchase a product that maximizes their expected utility.

There is a single trait, τ, with exogenous support, $\xi_\tau = \{T, C\}$. The trait has a default probability distribution with density function $\phi_\tau(\cdot \mid a)$ that assigns a probability of .5 to each of the two elements of the support. Each firm, f, chooses $\phi(\xi \mid a^f)$ for all $\xi \in \xi_\tau$. Movement away from the default distribution incurs a cost represented by:

$$\sum_{\xi \in \xi_\tau} c\left(\mid \phi(\xi \mid a^f) - \phi_\tau(\xi \mid a^f) \mid\right)$$

where $c(0) = 0$, $c'(0) = 0$, and $c''(x) > 0$ for all x. To guarantee uniqueness of equilibrium, we impose $c'''(x) \geq 0$ for all x. Firms simultaneously and

[16] Product characteristics are a prominent factor of many models of consumer demand. Among much literature regarding demand estimation, see Berry (1994), Berry, Levinsohn, and Pakes (1995), and Nevo (2001).

independently choose the density functions for each trait, observe the choice of density function of the other firm, then simultaneously and independently choose prices. The utility of type ζ from a^f is

$$u_\zeta(a^f) = \zeta\big[\phi(T \mid a^f) - \phi(C \mid a^f)\big] + \phi(C \mid a^f).$$

Here T and C can be thought of as two characteristics of the good that are substitutes in the sense that more of one implies less of the other. Utility is a weighted average of the T and C content in a good, where ζ indexes the preference for T over C. This framework differs significantly from Hotelling (1929) and d'Aspremont, Gabszewicz, and Thisse (1979). Rather than a continuum of ideal products associated with the continuum of consumers, all agents either prefer T or C, however are heterogeneous in their intensity of preference.

Consumers know the products, however are unable to distinguish them *ex ante*. Consumers that view the trait for a single product and view T, deduce (using Bayes' rule) that the product is a^i with probability $\frac{\phi(T|a^i)}{\phi(T|a^i)+\phi(T|a^j)}$ and that the product is a^j with probability $\frac{\phi(T|a^j)}{\phi(T|a^i)+\phi(T|a^j)}$.

Type ζ consumers then have expected utility

$$V_T^\zeta = \frac{\phi(T \mid a^i)u_\zeta(a^i) + \phi(T \mid a^j)u_\zeta(a^j)}{\phi(T \mid a^i) + \phi(T \mid a^j)}$$

from the product viewed and expected utility

$$V_{\sim T}^\zeta = \frac{\phi(T \mid a^i)u_\zeta(a^j) + \phi(T \mid a^j)u_\zeta(a^i)}{\phi(T \mid a^i) + \phi(T \mid a^j)}$$

from the other product. Similarly if viewing C, type ζ consumers have expected utility of

$$V_C^\zeta = \frac{\phi(C \mid a^i)u_\zeta(a^i) + \phi(C \mid a^j)u_\zeta(a^j)}{\phi(C \mid a^i) + \phi(C \mid a^j)}$$

for the product viewed and expected utility of

$$V_{\sim C}^{\zeta} = \frac{\phi(C \mid a^j)u_\zeta(a^i) + \phi(C \mid a^i)u_\zeta(a^j)}{\phi(C \mid a^i) + \phi(C \mid a^j)}$$

for the other product. Without loss of generality, assume $\phi(C|a^i) \geq \phi(C|a^j)$ throughout. A consumer of type ζ, which views T from firm j, will purchase the observed product if

$$V_T^{\zeta} - p_j > V_{\sim T}^{\zeta} - p_i$$

Plugging in for V_T^{ζ} and $V_{\sim T}^{\zeta}$ yields,

$$\zeta > \frac{1}{2} + \frac{(p_j - p_i)(\phi(T|a^j) + \phi(T|a^i))}{\left[\phi(T|a^j) - \phi(T|a^i)\right](\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i))} := \zeta_j^T$$

A consumer of type ζ that views C from firm j will purchase the observed product if

$$V_C^{\zeta} - p_j > V_{\sim C}^{\zeta} - p_i$$

which yields

$$\zeta < \frac{1}{2} + \frac{(p_j - p_i)(\phi(C|a^j) + \phi(C|a^i))}{\left[\phi(C|a^j) - \phi(C|a^i)\right](\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i))} := \zeta_j^C$$

There is a $\frac{1}{2}$ probability a consumer views the trait from a^j or a^i. The revenue function of each firm is:

$$r_j = p_j \left(\frac{1}{2} \left[\phi(T|a^j)(1 - \zeta_j^T) + \phi(C|a^j)\zeta_j^C + \phi(T|a^i)\zeta_i^T + \phi(C|a^i)(1 - \zeta_i^C)\right] \right)$$

$$r_i = p_i \left(\frac{1}{2} \left[\phi(T|a^j)\zeta_j^T + \phi(C|a^j)(1 - \zeta_j^C) + \phi(T|a^i)(1 - \zeta_i^T) + \phi(C|a^i)\zeta_i^C\right] \right).$$

For the purposes of comparison, note that if consumers are fully-rational, then the revenue functions are:[17]

$$r_j = p_j \left(\frac{1}{2} - \frac{p_j - p_i}{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)} \right)$$

[17] Fully rational agents purchase from j if $\zeta\left[\phi(T|a^j) - \phi(C|a^j)\right] + \phi(C|a^j) - p_j > \zeta\left[\phi(T|a^i) - \phi(C|a^i)\right] + \phi(C|a^i) - p_i$. This solves to $\zeta > \frac{1}{2} + \frac{p_j - p_i}{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)}$.

$$r_i = p_i \left(\frac{1}{2} + \frac{p_j - p_i}{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)} \right)$$

The profit functions are:

$$\Pi_j = r_j - c\left(\mid \phi(C \mid a^j) - \phi_\tau(C \mid a^j) \mid\right) - c\left(\mid \phi(T \mid a^j) - \phi_\tau(T \mid a^j) \mid\right)$$

$$\Pi_i = r_i - c\left(\mid \phi(C \mid a^i) - \phi_\tau(C \mid a^i) \mid\right) - c\left(\mid \phi(T \mid a^i) - \phi_\tau(T \mid a^i) \mid\right).$$

A strategy for firm i is a pair, $\{\phi(T|a^i), p_i(\phi(T|a^i), \phi(T|a^j))\}$ where $\phi(T|a^i)$ is a probability and $p_i : [0,1]^2 \to \mathbb{R}$. An equilibrium is a strategy pair such that

$$p_i \in \arg_p \max \Pi_i(\phi(T|a^i), \phi(T|a^j), p, p_j)$$

and

$$\phi(T|a^i) \in \arg_\phi \max \Pi_i \Big[\phi, \phi(T|a^j), p_i(\phi, \phi(T|a^j)), p_j(\phi, \phi(T|a^j))\Big].$$

A *symmetric* equilibrium is an equilibrium such that $\phi(T|a^i) = 1 - \phi(T|a^j)$ and $p_i = p_j$.

Note that the revenue functions (and hence the profit functions) are not defined where $\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i) = 0$. By defining revenues to be zero at such points, we create the continuous extensions of the profit functions. This is true whether consumers are of limited-skill or fully rational, as follows from lemma 3.4 and lemma 3.5 respectively.

Lemma 3.4 with limited-skill consumers first establishes the existence of mutual best reply prices given the trait choices of each firm. It is then shown that as the firms' traits become more similar, the mutual best reply prices converge to 0. Finally, if each firm chooses the same trait, then mutual best reply prices are 0. Lemma 3.5 establishes the analogous results with fully rational consumers.

Lemma 3.4 With limited-skill agents, given $\{\phi(T|a^i), \phi(T|a^j)\}$, there exist mutual best reply prices, $\{p_i(\phi(T|a^i), \phi(T|a^j)), p_j(\phi(T|a^i), \phi(T|a^j))\}$. Furthermore (1) $p_i(\phi(T|a^i), \phi(T|a^j)) = p_j(\phi(T|a^i), \phi(T|a^j))$,
(2) if $\phi(T|a^i) = \phi(T|a^j)$, then $p_i(\phi(T|a^i), \phi(T|a^j)) = 0$,
and (3) $\lim_{|\phi(T|a^i)-\phi(T|a^j)|\to 0} p_i(\phi(T|a^i), \phi(T|a^j)) = 0$.
Proof: See Appendix.

Lemma 3.5 With fully rational consumers, given $\{\phi(T|a^i), \phi(T|a^j)\}$, there exist mutual best reply prices, $\{p_i(\phi(T|a^i), \phi(T|a^j)), p_j(\phi(T|a^i), \phi(T|a^j))\}$. Furthermore (1) $p_i(\phi(T|a^i), \phi(T|a^j)) = p_j(\phi(T|a^i), \phi(T|a^j))$,
(2) if $\phi(T|a^i) = \phi(T|a^j)$, then $p_i(\phi(T|a^i), \phi(T|a^j)) = 0$,
and (3) $\lim_{|\phi(T|a^i)-\phi(T|a^j)|\to 0} p_i(\phi(T|a^i), \phi(T|a^j)) = 0$.

Proof: See Appendix.

With lemmata 3.4 and 3.5, the symmetric equilibria with limited-skill consumers and fully rational consumers are derived in proposition 3.6.

Proposition 3.6 For both limited-skill and fully rational agents, there exists a unique symmetric equilibrium. With limited-skill consumers equilibrium choices of $\phi(T|a^f)$ and $\phi(C|a^f)$ for $f \in \{i, j\}$ are implicitly defined by:

$$\phi(T|a^j) = c'(|\phi(T|a^j) - .5|) + \frac{1}{2}$$

$$\phi(C|a^j) = c'(|\phi(C|a^i) - .5|) + \frac{1}{2}$$

and with fully rational consumers equilibrium choices are implicitly defined by

$$\frac{1}{4} = c'(|\phi(T|a^j) - .5|)$$
$$\frac{1}{4} = c'(|\phi(C|a^i) - .5|).$$

Proof: See Appendix.

Notice that increasing $\phi(T|a^f)$ concurrently decreases $\phi(C|a^f)$. Therefore the equilibrium conditions with fully rational consumers are equivalent to:

$$\frac{1}{2} = c'(|\phi(T|a^j) - .5|) - c'(|\phi(C|a^j) - .5|)$$
$$\frac{1}{2} = c'(|\phi(C|a^i) - .5|) - c'(|\phi(T|a^i) - .5|)$$

Best response pricing always results in sales of $\frac{1}{2}$, and it follows from Lemma 3.5 that the derivative of mutual best reply prices with respect to $\phi(T|a^j)$ is 1. Therefore the derivative of marginal revenue with respect to $\phi(T|a^i)$ is $\frac{1}{2}$, namely, the equilibrium marginal cost of changing $\phi(T|a^f)$ and $\phi(C|a^f)$.

Recall it was assumed that $\phi(C|a^i) > \phi(C|a^j)$, therefore in a symmetric equilibrium, $\phi(T|a^j) = \phi(C|a^i) > \frac{1}{2} > \phi(T|a^i) = \phi(C|a^j)$. The main result of this section is that if marginal costs of further distinguishing are sufficiently low, equilibrium $\phi(T|a^j) - \phi(T|a^i)$ is greater with limited-skill consumers than with fully rational consumers. Let ϕ^R denote the equilibrium choice of $\phi(T|a^j)$ with fully rational consumers and ϕ^{LS} the equilibrium choice of $\phi(T|a^j)$ with limited-skill consumers.

Proposition 3.7 If $c'(1/4) \leq 1/4$, then $\phi^{LS} \geq \phi^R$. If $c'(1/4) \geq 1/4$, then $\phi^{LS} \leq \phi^R$.

Proof: See Appendix.

Intuitively, with fully rational consumers, product differentiation results in higher prices due to the increase in utility to consumers that purchase the product. Marginal utility to consumers and marginal revenues to firms due to product differentiation are constant. With limited-skill consumers, the marginal returns to further distinguishing a product are initially zero and everywhere increasing. When products are identical, the increase in price resulting from product differentiation is 0 because consumers are unable to distinguish the products and, to a first order approximation, gain no ability to do so. As products are increasingly differentiated, firms are able to charge higher prices because consumers are increasingly able to distinguish the products. For rapidly increasing marginal costs as product differentiation increases, the intersection of marginal costs and marginal revenue occurs with less differentiation with limited-skill consumers than with fully rational consumers.

We now turn to the welfare properties of equilibrium. The social planner maximizes the utility of consumers net of the cost of differentiating products. Profits are a transfer and ignored. Because of the inelasticity of demand, prices have no effect on welfare as long as they are equal to each other. Best response prices given $\phi(T|a^j)$ and $\phi(T|a^i)$ satisfy this property. Therefore the social planner only chooses the level of differentiation.

Fully Rational Consumers

Recall that, without loss of generality, $\phi(C|a^i) \geq \phi(C|a^j)$. Fully rational agents with $\zeta > .5$ purchase a^j and agents with $\zeta < .5$ purchase from a^i. Therefore, the social planner's problem with fully rational agents is:

$$\max_{\phi_i,\phi_j} W = \int_{.5}^{1} u_\zeta(a^j)d\zeta + \int_{0}^{.5} u_\zeta(a^i)d\zeta - 2c(|\phi(T|a^j) - .5|) - 2c(|\phi(T|a^i) - .5|)$$

One first order condition is

$$\frac{\partial W}{\partial \phi(T|a^j)} = \int_{.5}^{1} (2\zeta - 1)d\zeta - 2c'(|\phi(T|a^j) - .5|) = 0,$$

which implies

$$1/4 = 2c'(|\phi(T|a^j) - .5|).$$

The symmetry of the problem implies the other first order condition is

$$1/4 = 2c'(|\phi(C|a^i) - .5|).$$

The second order condition always holds by convexity of costs. Comparing these first order conditions with the equilibrium conditions from proposition 3.6, establishes that equilibrium product differentiation is greater than socially optimal product differentiation.

Limited-Skill Consumers

With limited-skill agents, the social planner's problem must take into account the probabilities that an agent purchases each product:

$$\max_{\phi_i,\phi_j} W = \int_0^1 \left[u_\zeta(a^j) Pr(a^j|\zeta) + u_\zeta(a^i) Pr(a^i|\zeta) \right] d\zeta$$

$$-2c(|\phi(T|a^j) - .5|) - 2c(|\phi(T|a^i) - .5|)$$

where $Pr(a^j|\zeta)$ is the probability of purchasing a^j given utility maximizing behavior. If $\zeta > \frac{1}{2}$ this is:

$$\frac{1}{2}\left[\phi(T|a^j) + \phi(C|a^i) \right].$$

If $\zeta < \frac{1}{2}$ the probability of purchasing a^j is:

$$\frac{1}{2}\left[\phi(C|a^j) + \phi(T|a^i) \right].$$

with analogous expressions for purchasing a^i.

The derivation of first order conditions is straightforward, but is tedious and relegated to the appendix. The conditions for maximizing social welfare are:

$$\phi(T|a^j) = \frac{1}{2} + 2c'(|\phi(T|a^j) - .5|)$$
$$\phi(C|a^i) = \frac{1}{2} + 2c'(|\phi(C|a^i) - .5|).$$

Notice $\phi(T|a^j) = \phi(C|a^i) = .5$ satisfies these equations, however only maximizes social welfare if the second derivative of welfare with respect to differentiating each good is negative at such a point. The condition is $c''(0) \geq 1/4$ and derived in the appendix. Any other values of $\phi(T|a^j)$ and $\phi(C|a^i)$ which satisfy the conditions are unique due to $c'''(x) \geq 0$ for all x.

As with fully rational consumers, comparing these first order conditions with the equilibrium conditions from proposition 3.6, establishes that equilibrium product differentiation is greater than socially optimal product differentiation.

The heuristic explanation for equilibrium failing to exhibit socially optimal behavior results from market power over a subset of consumers which is enhanced by product differentiation. To see this, consider the effect of increasing $\phi(T|a^j)$. This increases $u_\zeta(a^j) - u_\zeta(a^i)$ if $\zeta > .5$, allowing for p_j to increase. This also causes $u_\zeta(a^i) - p_i - (u_\zeta(a^j) - p_j)$ to increase for consumers with $\zeta < .5$. This allows firm i to increase p_i and still garner the same number of sales, a result of market power gained through the increase in differentiation of product j. As p_i increases, $u_\zeta(a^j) - p_j - (u_\zeta(a^i) - p_i)$ increases for consumers with $\zeta > .5$, allowing for firm j to increase p_j. Thus firms have two incentives to increase product differentiation (1) an incentive to increase differentiation to garner market power, (2) an incentive to increase utility of consumers to support a price increase. A social planner only has an incentive to increase differentiation to increase utility of consumers.

4 Concluding remarks

This paper has developed a theoretical model where agents are able to consider only a subset of available actions. It seems plausible that this mimics the way individuals manage their cognitive limitations while taking into account the cognitive limitations of others. This framework is sufficiently tractable to be grafted into many applications, producing insights that don't arise in the fully rational case. It also makes explicit the underlying exogenous causes of bounded rationality. Future research may uncover additional applications and generate more results within the structure of limited-skill agents and trait-augmented games.

References

Abdulkadiroglu, A., Agarwal, N., & Pathak, P. A. (2017). The welfare effects of coordinated assignment: Evidence from the New York City high school match. *American Economic Review*, 107(12), 3635-89.

Abreu, D., and Rubinstein, A. (1988). The Structure of Nash Equilibrium in Repeated Games with Finite Automata. *Econometrica*, 56(6) pp. 1259-1281.

Apicella, C. L., Azevedo, E. M., Christakis, N. A., & Fowler, J. H. (2014). Evolutionary origins of the endowment effect: evidence from hunter-gatherers. *The American Economic Review*, 104(6), 1793-1805.

Augenblick, N. (2016). The sunk-cost fallacy in penny auctions. *The Review of Economic Studies*, 83(1), 58-86.

Azevedo, E. M., & Leshno, J. D. (2016). A supply and demand framework for two-sided matching markets. *Journal of Political Economy*, 124(5), 1235-1268.

Berry, S. T. (1994). Estimating discrete-choice models of product differentiation. *The RAND Journal of Economics*, 242-262.

Berry, S., Levinsohn, J., & Pakes, A. (1995). Automobile prices in market equilibrium. *Econometrica*, 841-890.

Chen, Y., Su, X., & Zhao, X. (2012). Modeling bounded rationality in capacity allocation games with the quantal response equilibrium. *Management Science*, 58(10), 1952-1962.

Chetty, R., Looney, A., & Kroft, K. (2009). Salience and taxation: Theory and evidence. *The American economic review*, 99(4), 1145-1177.

d'Aspremont, C., Gabszewicz, J. J., & Thisse, J. F. (1979). On Hotelling's"

Stability in competition". *Econometrica*, 1145-1150.

Fudenberg, D., & Levine, D. K. (1993). Self-confirming equilibrium. *Econometrica*, 523-545.

Fox, J. T. (2018). Estimating matching games with transfers. *Quantitative Economics*, 9(1), 1-38.

Gabaix, X. (2014). A sparsity-based model of bounded rationality. The *Quarterly Journal of Economics*, 129(4), 1661-1710.

Gale, D., & Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1), 9-15.

Goldin, J. (2015). Optimal tax salience. *Journal of Public Economics*, 131, 115-123.

Harstad, R. M., & Selten, R. (2013). Bounded-rationality models: tasks to become intellectually competitive. *Journal of Economic Literature*, 51(2), 496-511.

Hotelling, H. (1929). Stability in competition, *Economic Journal* 39, 41-57.

Jéheil, P. (1995) Limited horizon forecast in repeated alternate games. *Journal of Economic Theory*, 67 (2) (1995), pp. 497–519

Jéheil P., Samet D., 2005 Learning to play games in extensive form by valuation. *Journal of Economic Theory*, 124 (2005), pp. 129–148

Kagel, J. H., & Roth, A. E. (2000). The dynamics of reorganization in matching markets: A laboratory experiment motivated by a natural experiment. *The Quarterly Journal of Economics*, 115(1), 201-235.

Lambson, V., and van den Berghe J. "Skill, complexity, and strategic interaction." *Journal of Economic Theory* 159 (2015): 516-530.

Lambson, V. E., & Probst, D. A. (2004). Learning by matching patterns. *Games and Economic Behavior*, 46(2), 398-409.

Loewenstein, G., Rabin, M., & Camerer, C. (2004). Advances in behavioral economics. Russell, New York.

McKelvey, R. D., & Palfrey, T. R. (1995). Quantal response equilibria for normal form games. *Games and economic behavior*, 10(1), 6-38.

McKelvey, R. D., & Palfrey, T. R. (1998). Quantal response equilibria for extensive form games. *Experimental economics*, 1(1), 9-41.

Monte, D. (2014). Learning with bounded memory in games. *Games and Economic Behavior*, 87, 204-223.

Nevo, A. (2001). Measuring market power in the ready-to-eat cereal industry. *Econometrica*, 69(2), 307-342.

Roth, A. E., & Erev, I. (1995). Learning in extensive-form games: Experimental data and simple dynamic models in the intermediate term. *Games and Economic Behavior*, 8(1), 164-212.

Roth, A. E., & Sotomayor, M. A. (1990). Two-sided matching: A study in game-theoretic modeling and analysis. Cambridge: Cambridge University Press.

Samuelson, L. (1998). *Evolutionary games and equilibrium selection* (Vol. 1). MIT press.

Saran, R. (2016). Bounded depths of rationality and implementation with complete information. *Journal of Economic Theory*, 165, 517-564.

Simon, H. A. (1957). *Models of Man Social and Rational, Mathematical Essays on Rational Human Behavior in a Social Setting. Herbert A. Simon,...* J. Wiley and Sons.

Sip, K. E., Smith, D. V., Porcelli, A. J., Kar, K., & Delgado, M. R. (2015). Social closeness and feedback modulate susceptibility to the framing effect. *Social neuroscience*, 10(1), 35-45.

Appendix: Proofs

Proof of Proposition 3.3

Consider g_1, g_2, b_1, b_2 such that $V_{g_1}(g_1, b_1) > V_{g_1}(g_1, b_2)$ and $V_{b_1}(g_1, b_1) > V_{b_1}(g_2, b_1)$. If $(g_2, b_1) \in P$ and $(g_1, b_2) \in P$, then g_1 and b_1 are mismatched. Let m_b be the move associated with the action of proposing to b, and m_g the move associated with the action of accepting a proposal from g.

We first show that g_1 proposes to b_2 with positive probability. (Similar reasoning establishes that g_2 proposes to b_1 with positive probability). Let η_{g_1} be the consideration set chosen by g_1 in the first round and θ the realized observation set. Trivially, there exists some $m_{\bar{b}}$ for which

$$\sum_{b \in B} V_{g_1}(g_1, b)\mu(b|\theta, m_{\bar{b}}) \geq \sum_{b \in B} V_{g_1}(g_1, b)\mu(b|\theta, m_{b^*})$$

for all $b^* \in B$. Define $\eta^2_{g_1}$ as follows. If $(m_b, \tau) \in \eta_{g_1}$, $b \neq b_2, b \neq \bar{b}$, for some τ, then $(m_b, \tau) \in \eta^2_{g_1}$. If $(m_{b_2}, \tau) \in \eta_{g_1}$, for some τ, then $(m_{\bar{b}}, \tau) \in \eta^2_{g_1}$. If $(m_{\bar{b}}, \tau) \in \eta_{g_1}$, for some τ, then $(m_{b_2}, \tau) \in \eta^2_{g_1}$.[18]

It was assumed the support of each trait is the same for all agents, therefore the probability of viewing all the same observations from $\eta^2_{g_1}$ is positive. Because g_1 is unable to *ex ante* distinguish the moves, g_1 is equally likely to choose $\eta^2_{g_1}$ as choose η_{g_1}. Therefore there is a positive probability of choosing $\eta^2_{g_1}$ and an observation set being realized which results in:

$$\sum_{b \in B} V_{g_1}(g_1, b)\mu(b|\theta, m_{b_2}) \geq \sum_{b \in B} V_{g_1}(g_1, b)\mu(b|\theta, m_{b^*})$$

[18]In words, we replace $m_{\bar{b}}$ with m_{b_2} and replace m_{b_2} with $m_{\bar{b}}$, and leave all else the same.

This induces g_1 to propose to b_2 with positive probability.

We next show that, if g_1 proposes to b_2, then b_2 accepts the proposal from g_1 with positive probability. (Similar reasoning establishes that, if g_2 proposes to b_1, then b_1 accepts the proposal from g_2 with positive probability). Regardless of the other proposals received by b_2, for the η_{b_2} chosen by b_2 in the first round and the realized θ, there exists an equally likely $\eta^2_{b_2}$ which is chosen with positive probability from which the same observations are realized such that $\sum\limits_{g \in G} V_{b_2}(g, b_2)\mu(g|\theta, m_g) \geq \sum\limits_{g \in G} V_{b_2}(g, b_2)\mu(g|\theta, m_{g^*})$ for all g^* which have proposed. This follows by similar reasoning to that above. Now, b_2 will accept g_1's proposal with positive probability.

Finally we show that, with positive probability, b_2 and b_1 will receive no more proposals. In all future rounds, all $g \in G$, $g \neq g_1$, and $g \neq g_2$ will, with positive probability, choose consideration sets and realize observation sets which induce them to propose to $b \in B$, $b \neq b_1$, $b \neq b_2$. This follows by the same reasoning as above, noting that all $g \in G$ forget traits at the end of each round. Because $\#G = \#B$, all $g \in G$ will match without b_1 or b_2 receiving another proposal. Therefore, with positive probability b_2 and b_1 will never receive another proposal. Thus (g_1, b_2) and (g_2, b_1) are partnerships which occur with positive probability.

QED

Proof of Lemma 3.4

Recall:

$$r_j = p_j \left(\frac{1}{2} \left[\phi(T|a^j)(1 - \zeta_j^T) + \phi(C|a^j)\zeta_j^C + \phi(T|a^i)\zeta_i^T + \phi(C|a^i)(1 - \zeta_i^C) \right] \right)$$

The derivative of profit with respect to price is:

$$\frac{\partial \Pi_j}{\partial p_j} = \left(\frac{1}{2}\left[\phi(T|a^j)(1-\zeta_j^T) + \phi(C|a^j)\zeta_j^C + \phi(T|a^i)\zeta_i^T + \phi(C|a^i)(1-\zeta_i^C)\right]\right)$$
$$+ p_j\frac{1}{2}\Bigg(\phi(T|a^j)\frac{-(\phi(T|a^j)+\phi(T|a^i))}{\left[\phi(T|a^j)-\phi(T|a^i)\right](\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}$$
$$+\phi(C|a^j)\frac{(\phi(C|a^j)+\phi(C|a^i))}{\left[\phi(C|a^j)-\phi(C|a^i)\right](\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}$$
$$+\phi(T|a^i)\frac{-(\phi(T|a^j)+\phi(T|a^i))}{\left[\phi(T|a^j)-\phi(T|a^i)\right](\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}$$
$$+\phi(C|a^i)\frac{(\phi(C|a^j)+\phi(C|a^i))}{\left[\phi(C|a^j)-\phi(C|a^i)\right](\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}\Bigg) = 0$$

Multiplying each side by $2(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))$ results in:

$$0 = \Bigg(\Bigg[\phi(T|a^j)\left(\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2} - \frac{(p_j-p_i+p_j)(\phi(T|a^j)+\phi(T|a^i))}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right)$$
$$+\phi(C|a^j)\left(\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2} + \frac{(p_j-p_i+p_j)(\phi(C|a^j)+\phi(C|a^i))}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)$$
$$+\phi(T|a^i)\left(\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2} - \frac{(p_j-p_i+p_j)(\phi(T|a^j)+\phi(T|a^i))}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right)$$
$$+\phi(C|a^i)\left(\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2} + \frac{(p_j-p_i+p_j)(\phi(C|a^j)+\phi(C|a^i))}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)\Bigg]\Bigg)$$

Solving for p_j

$$p_j = \Bigg(\Bigg[\phi(T|a^j)\left(\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2} - \frac{(-p_i)(\phi(T|a^j)+\phi(T|a^i))}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right)$$
$$+\phi(C|a^j)\left(\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2} + \frac{(-p_i)(\phi(C|a^j)+\phi(C|a^i))}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)$$

$$+\phi(T|a^i)\left(\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2}-\frac{(-p_i)(\phi(T|a^j)+\phi(T|a^i))}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right)$$

$$+\phi(C|a^i)\left(\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2}+\frac{(-p_i)(\phi(C|a^j)+\phi(C|a^i))}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)\Bigg]\Bigg)\Bigg/$$

$$2\Bigg[\phi(T|a^j)\left(\frac{(\phi(T|a^j)+\phi(T|a^i))}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right)-\phi(C|a^j)\left(\frac{(\phi(C|a^j)+\phi(C|a^i))}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)$$

$$+\phi(T|a^i)\left(\frac{(\phi(T|a^j)+\phi(T|a^i))}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right)-\phi(C|a^i)\left(\frac{(\phi(C|a^j)+\phi(C|a^i))}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)\Bigg]$$

Which simplifies to

$$p_j=\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2\left[\left(\frac{(\phi(T|a^j)+\phi(T|a^i))^2}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right)-\left(\frac{(\phi(C|a^j)+\phi(C|a^i))^2}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)\right]}+\frac{p_i}{2}$$

A similar derivation shows that

$$p_i=\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{2\left[\left(\frac{(\phi(T|a^j)+\phi(T|a^i))^2}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right)-\left(\frac{(\phi(C|a^j)+\phi(C|a^i))^2}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)\right]}+\frac{p_j}{2}$$

The mutual best reply prices are then

$$p_j=p_i=\frac{(\phi(T|a^j)-\phi(C|a^j)-\phi(T|a^i)+\phi(C|a^i))}{\left(\frac{(\phi(T|a^j)+\phi(T|a^i))^2}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right)-\left(\frac{(\phi(C|a^j)+\phi(C|a^i))^2}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)}$$

Notice that p_j and p_i can be simplified as follows because $\phi(C|a^i) = 1-\phi(T|a^i)$:

$$p_j=p_i=\frac{(2\phi(T|a^j)-2\phi(T|a^i))}{\left(\frac{(\phi(T|a^j)+\phi(T|a^i))^2}{\phi(T|a^j)-\phi(T|a^i)}\right)+\left(\frac{(2-\phi(T|a^j)-\phi(T|a^i))^2}{\phi(T|a^j)-\phi(T|a^i)}\right)}$$

$$p_j=p_i=\frac{2(\phi(T|a^j)-\phi(T|a^i))^2}{(\phi(T|a^j)+\phi(T|a^i))^2+(2-\phi(T|a^j)-\phi(T|a^i))^2}$$

That $p_j = p_i = 0$ when $\phi(T|a^j) - \phi(T|a^i) = 0$ is immediate. Similarly it is immediate that if $|\phi(T|a^j) - \phi(T|a^i)| \to 0$ then $p_j = p_i \to 0$.

The second derivative of profit with respect to price is:

$$\frac{\partial^2 \Pi_j}{\partial p_j^2} = \Bigg(\phi(T|a^j) \frac{-(\phi(T|a^j) + \phi(T|a^i))}{\big[\phi(T|a^j) - \phi(T|a^i)\big](\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i))}$$

$$+ \phi(C|a^j) \frac{(\phi(C|a^j) + \phi(C|a^i))}{\big[\phi(C|a^j) - \phi(C|a^i)\big](\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i))}$$

$$+ \phi(T|a^i) \frac{-(\phi(T|a^j) + \phi(T|a^i))}{\big[\phi(T|a^j) - \phi(T|a^i)\big](\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i))}$$

$$+ \phi(C|a^i) \frac{(\phi(C|a^j) + \phi(C|a^i))}{\big[\phi(C|a^j) - \phi(C|a^i)\big](\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i))} \Bigg)$$

By assumption without loss of generality, $\phi(C|a^j) \leq \phi(C|a^i)$. Therefore each term of the second derivative is strictly negative except at the points of the continuous extension, where the second derivative is undefined.

QED

Proof of Lemma 3.5

Recall

$$r_j = p_j \left(\frac{1}{2} - \frac{p_j - p_i}{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)} \right)$$

Consider the best response price of firm j, determined by $\frac{\partial \Pi_j}{\partial p_j} = 0$,

$$0 = \left(\frac{1}{2} - \frac{p_j - p_i}{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)} \right) - \left(\frac{p_j}{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)} \right)$$

$$p_j = \frac{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)}{4} + \frac{p_i}{2}$$

Similarly,

$$p_i = \frac{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)}{4} + \frac{p_j}{2}$$

Best response pricing is then

$$p_j = p_i = \frac{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)}{2}$$

Noting that $\phi(T|a^i) = 1 - \phi(C|a^i)$,

$$p_j = p_i = \phi(T|a^j) - \phi(T|a^i)$$

That $p_j = p_i = 0$ when $\phi(T|a^j) - \phi(T|a^i) = 0$ is immediate. Similarly it is immediate that if $|\phi(T|a^j) - \phi(T|a^i)| \to 0$ then $p_j = p_i \to 0$.

The second order condition is

$$\frac{\partial^2 \Pi_j}{\partial p_j^2} = 0 = -\frac{2p_j}{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)} \leq 0$$

By assumption without loss of generality, $\phi(C|a^j) \leq \phi(C|a^i)$. Again the second derivative is undefined only at the points of the continuous extension, and strictly negative at all other points.

QED

Proof of Proposition 3.6

Limited-Skill Agents

By lemma 3.4, for any $\phi(T|a^j)$ and $\phi(T|a^i)$ and best-response pricing, all $\zeta_f^T, \zeta_f^C = \frac{1}{2}$ for $f \in \{i, j\}$. Furthermore the derivative of these zeta terms with respect to $\phi(T|a^i)$ and $\phi(T|a^j)$ is 0 by an application of the chain rule and still imposing best-reply pricing. Plugging in for p_j and noting that quantity sold is always $\frac{1}{2}$ due to all ζ terms being $\frac{1}{2}$, profit for firm j can be written.

$$\Pi_j = \frac{1}{2}\frac{(\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i))}{\left(\frac{(\phi(T|a^j)+\phi(T|a^i))^2}{\left[\phi(T|a^j)-\phi(T|a^i)\right]}\right) - \left(\frac{(\phi(C|a^j)+\phi(C|a^i))^2}{\left[\phi(C|a^j)-\phi(C|a^i)\right]}\right)} - c(|\phi(C|a^j) - .5|) - c(|\phi(T|a^j) - .5|)$$

Writing $\phi(C|a^f) = 1 - \phi(T|a^f)$:

$$\Pi_j = \frac{1}{2}\frac{(2\phi(T|a^j) - 2\phi(T|a^i))}{\left(\frac{(\phi(T|a^j)+\phi(T|a^i))^2}{\phi(T|a^j)-\phi(T|a^i)}\right) + \left(\frac{(2-\phi(T|a^j)-\phi(T|a^i))^2}{\phi(T|a^j)-\phi(T|a^i)}\right)} - 2c(|\phi(T|a^j) - .5|)$$

Simplifying:

$$\Pi_j = \frac{(\phi(T|a^j) - \phi(T|a^i))^2}{(\phi(T|a^j) + \phi(T|a^i))^2 + (2 - \phi(T|a^j) - \phi(T|a^i))^2} - 2c(|\phi(T|a^j) - .5|)$$

$$\Pi_j = \frac{\phi(T|a^j)^2 + \phi(T|a^i)^2 - 2\phi(T|a^i)\phi(T|a^j)}{\phi(T|a^j)^2 + \phi(T|a^i)^2 + 2\phi(T|a^i)\phi(T|a^j) + (2 - \phi(T|a^j) - \phi(T|a^i))^2} - 2c(|\phi(T|a^j) - .5|)$$

$$\Pi_j = \frac{\phi(T|a^j)^2 + \phi(T|a^i)^2 - 2\phi(T|a^i)\phi(T|a^j)}{4 + 2\phi(T|a^j)^2 + 2\phi(T|a^i)^2 + 4\phi(T|a^i)\phi(T|a^j) - 4\phi(T|a^j) - 4\phi(T|a^i)} - 2c(|\phi(T|a^j) - .5|)$$

Consider the first-order condition:

$$\frac{\partial \Pi_j}{\partial \phi(T|a^j)}$$
$$= \Big(\left[4 + 2\phi(T|a^j)^2 + 2\phi(T|a^i)^2 + 4\phi(T|a^i)\phi(T|a^j) - 4\phi(T|a^j) - 4\phi(T|a^i)\right]\left[2\phi(T|a^j) - 2\phi(T|a^i)\right]$$
$$- \left[\phi(T|a^j)^2 + \phi(T|a^i)^2 - 2\phi(T|a^i)\phi(T|a^j)\right]\left[4\phi(T|a^j) + 4\phi(T|a^i) - 4\right]\Big)/$$
$$\left(4 + 2\phi(T|a^j)^2 + 2\phi(T|a^i)^2 + 4\phi(T|a^i)\phi(T|a^j) - 4\phi(T|a^j) - 4\phi(T|a^i)\right)^2 - 2c'(|\phi(T|a^j) - .5|) = 0$$

By the symmetry of the problem an identical expression exists for firm i. Solving for the equilibrium choices of $\phi(T|a^j)$ and $\phi(C|a^i)$ is tedious but straightforward,

$$\phi(T|a^j) = c'(|\phi(T|a^j) - .5|) + \frac{1}{2}$$

$$\phi(C|a^i) = c'(|\phi(C|a^i) - .5|) + \frac{1}{2}$$

Or, if $\phi(T|a^j) > c'(|\phi(T|a^j) - .5|) + \frac{1}{2}$ for all $\phi(T|a^j) \in [.5, 1]$, there is a corner solution where $\phi(T|a^j) = \phi(C|a^i) = 1$. This occurs if and only if $c'(.5) < .5$.

The point $\phi(T|a^j) = \phi(C|a^i) = .5$ does satisfy the equilibrium condition, however returns profits of 0 to each firm and is only an equilibrium depending on the second derivative of profits with respect to differentiation, shown below.

Because $c'(0) = 0$ and $c'''(x) \geq 0$ for all x, there is at most one point other than $\phi(T|a^j) = .5$ where the first order condition holds. The equilibrium is that other point, if it exists, or there is a corner solution as described above.

The second order condition is checked as follows.

$$\begin{aligned}
&\frac{\partial^2 \Pi_j}{\partial \phi(T|a^j)^2} \\
&= \frac{2}{(2\phi(T|a^j)^2 + 4\phi(T|a^j)\phi(T|a^i) - 4\phi(T|a^j) + 2\phi(T|a^i)^2 - 4\phi(T|a^i) + 4)} \\
&- \frac{4(\phi(T|a^j)^2 - 2\phi(T|a^j)\phi(T|a^i) + \phi(T|a^i)^2)}{(2\phi(T|a^j)^2 + 4\phi(T|a^j)\phi(T|a^i) - 4\phi(T|a^j) + 2\phi(T|a^i)^2 - 4\phi(T|a^i) + 4)^2} \\
&- \frac{2(4\phi(T|a^j) + 4\phi(T|a^i) - 4)(2\phi(T|a^j) - 2\phi(T|a^i))}{(2\phi(T|a^j)^2 + 4\phi(T|a^j)\phi(T|a^i) - 4\phi(T|a^j) + 2\phi(T|a^i)^2 - 4\phi(T|a^i) + 4)^2} \\
&+ \frac{2(4\phi(T|a^j) + 4\phi(T|a^i) - 4)^2(\phi(T|a^j)^2 - 2\phi(T|a^j)\phi(T|a^i) + \phi(T|a^i)^2)}{(2\phi(T|a^j)^2 + 4\phi(T|a^j)\phi(T|a^i) - 4\phi(T|a^j) + 2\phi(T|a^i)^2 - 4\phi(T|a^i) + 4)^3} \\
&- 2c''(|\phi(T|a^j) - .5|)
\end{aligned}$$

When evaluated at $\phi(T|a^j) = \phi(C|a^i) = .5$ the second derivative is: $1 - 2c''(|\phi(T|a^j) - .5|)$. Therefore such a point is an equilibrium if and only if $c''(0) \geq \frac{1}{2}$. When evaluated at $\phi(T|a^j) = \phi(C|a^i) = \frac{1}{2} + c'(|\phi(T|a^j) - .5|) =$

$\frac{1}{2} + c'(|\phi(C|a^i) - .5|)$, the second derivative is:

$$\frac{\partial^2 \Pi_j}{\partial \phi(T|a^j)^2} = 1 - 4c'(\phi(T|a^j))^2 - 2c''(|\phi(T|a^j) - .5|)$$

With the assumptions that $c'(0) = 0$, and that $c''(x) > 0$ and $c'''(x) > 0$ for all x, it follows that $c''(x) > c'(x)/x$. Noting that $c'(\phi(T|a^j) = \phi(T|a^j) - \frac{1}{2}$ defines equilibrium, it is straightforward to verify the second order condition holds and the equation is negative.

$$\frac{\partial^2 \Pi_j}{\partial \phi(T|a^j)^2} \leq 1 - 4\left(\phi(T|a^j) - \frac{1}{2}\right)^2 - 2\frac{\phi(T|a^j) - \frac{1}{2}}{\phi(T|a^j) - \frac{1}{2}} < 0$$

Fully Rational Consumers

With fully rational consumers under best reply pricing, by lemma 3.5 $p_j = p_i$ and the revenue function is $p_j \frac{1}{2}$. Plugging in the best reply price from lemma 3.5, the profit function is:

$$\Pi_j = \frac{1}{2}\frac{\phi(T|a^j) - \phi(C|a^j) - \phi(T|a^i) + \phi(C|a^i)}{2} - c(|\phi(C|a^j) - .5|) - c(|\phi(T|a^j) - .5|)$$

$$\Pi_j = \frac{1}{2}\frac{2\phi(T|a^j) - 2\phi(T|a^i)}{2} - 2c(|\phi(T|a^j) - .5|)$$

$$\Pi_j = \frac{\phi(T|a^j) - \phi(T|a^i)}{2} - 2c(|\phi(T|a^j) - .5|)$$

The first order condition.

$$\frac{\partial \Pi_j}{\partial \phi(T|a^j)} = \frac{1}{2} - 2c'(|\phi(T|a^j) - .5|) = 0$$

By symmetry of the problem an analogous condition holds for firm i, the equilibrium is defined by $\phi(T|a^j)$ and satisfies

$$\frac{1}{4} = c'(|\phi(T|a^j) - .5|)$$

$$\frac{1}{4} = c'(|\phi(C|a^i) - .5|)$$

Or, if $\frac{1}{4} > c'(|\phi(T|a^j) - .5|)$ for all $\phi(T|a^j) \in [.5, 1]$, there is a corner solution where $\phi(T|a^j) = \phi(C|a^i) = 1$.

Uniqueness of equilibrium follows from monotonicity of $c'(x)$.

QED

Proof of Proposition 3.7

By Proposition 3.6, with fully rational consumers, equilibrium satisfies

$$c'(|\phi(T|a^j) - .5|) - \frac{1}{4} = 0$$

(Or there is a corner solution where $\phi(T|a^j) = \phi(C|a^i) = 1$.) Denote the equilibrium value of $\phi(T|a^j)$ with fully rational consumers as ϕ^R. With limited-skill agents equilibrium satisfies,

$$\phi(T|a^j) - c'(|\phi(T|a^j) - .5|) - \frac{1}{2} = 0 \qquad (*)$$

(Or there is a corner solution where $\phi(T|a^j) = \phi(C|a^i) = 1$, or where $\phi(T|a^j) = \phi(C|a^i) = .5$.) Denote the equilibrium value of $\phi(T|a^j)$ with limited-skill consumers as ϕ^{LS}.

Case 1: Corner Solutions.

If there is a corner solution with limited-skill agents where $\phi^{LS} = 1$, then $c'(1/2) < 1/2$ which implies $c'(1/4) \leq 1/4$ by $c'(0) = 0$, $c''(x) > 0$, and $c'''(x) \geq 0$ for all x. Trivially ϕ^{LS} is weakly greater than ϕ^R because ϕ^{LS} is maximized.

If there is a corner solution with fully rational consumers where $\phi^R = 1$, then $c'(1/2) < 1/4$ which implies $c'(1/4) < 1/4$. This also implies a corner solution with limited-skill agents because $c'(1/2) < 1/2$. Again ϕ^{LS} is weakly greater than ϕ^R and $c'(1/4) \leq 1/4$.

If there is a corner solution with limited-skill agents where $\phi^{LS} = .5$, then by proposition 3.6 it must be that $c''(0) \geq .5$. Because $c'''(x) \geq 0$ for all x, this implies $c'(1/4) \geq 1/4$. Also ϕ^{LS} is trivially lower than ϕ^R because ϕ^{LS} is minimized.

Case 2: No Corner Solutions.

Recall $c'''(x) \geq 0$ for all x. The key to the argument is that the derivative of the left-hand side of $(*)$ with respect to $\phi(T|a^j)$ is $1 - c''(|\phi(T|a^j) - .5|)$, and therefore the left hand side is increasing at $\phi(T|a^j) = .5$, and then will only cross 0 once (at the equilibrium).

If $c'(1/4) < 1/4$, then $\phi^R > 3/4$. If ϕ^R were plugged into (*), the left-hand side would return $\phi^R - 3/4 > 0$. Therefore $\phi^R < \phi^{LS}$.

If $c'(1/4) > 1/4$, then $\phi^R < 3/4$. If ϕ^R were plugged into (*), the left-hand side would return $\phi^R - 3/4 < 0$. Therefore $\phi^R > \phi^{LS}$.

If $c'(1/4) = 1/4$, then $\phi(T|a^j) = 3/4$ represents equilibrium in each market.

QED

Derivation of Welfare Maximizing Conditions

The case of fully rational agents is presented in the text, here the derivation for limited-skill agents is presented.

The welfare function is:

$$W = \int_0^{.5} \left[u_\zeta(a^j) Pr(a^j|\zeta) + u_\zeta(a^i) Pr(a^i|\zeta) \right] d\zeta +$$

$$\int_{.5}^{1} \left[u_\zeta(a^j)Pr(a^j|\zeta) + u_\zeta(a^i)Pr(a^i|\zeta)\right] d\zeta - 2c(|\phi(T|a^j) - .5|) - 2c(|\phi(T|a^i) - .5|) = 0.$$

One first order condition is:

$$\frac{\partial W}{\partial \phi(T|a^j)} = \int_{0}^{.5} (2\zeta - 1)Pr(a^j|\zeta) - u_\zeta(a^j)\frac{1}{2} + u_\zeta(a^i)\frac{1}{2}d\zeta + \int_{.5}^{1} (2\zeta - 1)Pr(a^j|\zeta) + u_\zeta(a^j)\frac{1}{2} - u_\zeta(a^i)\frac{1}{2}d\zeta$$

$$-2c'(|\phi(T|a^j) - .5|) = 0.$$

Plugging in for $Pr(a^j|\zeta)$:

$$\int_{0}^{.5} (2\zeta - 1)\frac{1}{2}\left[1 - \phi(T|a^j) + \phi(T|a^i)\right] - u_\zeta(a^j)\frac{1}{2} + u_\zeta(a^i)\frac{1}{2}d\zeta +$$

$$\int_{.5}^{1} (2\zeta - 1)\frac{1}{2}\left[\phi(T|a^j) + 1 - \phi(T|a^i)\right] + u_\zeta(a^j)\frac{1}{2} - u_\zeta(a^i)\frac{1}{2}d\zeta - 2c'(|\phi(T|a^j) - .5|) = 0.$$

Evaluating the "-1" portion of the $(2\zeta - 1)$ terms:

$$\int_{0}^{.5} \zeta\left[1 - \phi(T|a^j) + \phi(T|a^i)\right] - u_\zeta(a^j)\frac{1}{2} + u_\zeta(a^i)\frac{1}{2}d\zeta +$$

$$\int_{.5}^{1} \zeta\left[\phi(T|a^j) + 1 - \phi(T|a^i)\right] + u_\zeta(a^j)\frac{1}{2} - u_\zeta(a^i)\frac{1}{2}d\zeta - 2c'(|\phi(T|a^j) - .5|) = \frac{1}{2}.$$

Plugging in for $u_\zeta(a^f)$:

$$\int_{0}^{.5} \zeta\left[1 - \phi(T|a^j) + \phi(T|a^i)\right] - (\zeta(2\phi(T|a^j) - 1) + 1 - \phi(T|a^j))\frac{1}{2} + (\zeta(2\phi(T|a^i) - 1) + 1 - \phi(T|a^i))\frac{1}{2}d\zeta +$$

$$\int_{.5}^{1} \zeta\left[\phi(T|a^j) + 1 - \phi(T|a^i)\right] + (\zeta(2\phi(T|a^j) - 1) + 1 - \phi(T|a^j))\frac{1}{2} - (\zeta(2\phi(T|a^i) - 1) + 1 - \phi(T|a^i))\frac{1}{2}d\zeta$$

$$-2c'(|\phi(T|a^j) - .5|) = \frac{1}{2}.$$

Simplifying terms which evaluate to 0.

$$\int_0^{.5} \zeta\left[1 - \phi(T|a^j) + \phi(T|a^i)\right] - \zeta\phi(T|a^j) + \zeta\phi(T|a^i)d\zeta +$$

$$\int_{.5}^{1} \zeta\left[\phi(T|a^j) + 1 - \phi(T|a^i)\right] + \zeta\phi(T|a^j) - \zeta\phi(T|a^i)d\zeta - 2c'(|\phi(T|a^j) - .5|) = \frac{1}{2}.$$

Combining terms:

$$\int_0^{.5} \zeta\left[1 - 2\phi(T|a^j) + 2\phi(T|a^i)\right] d\zeta + \int_{.5}^{1} \zeta\left[2\phi(T|a^j) + 1 - 2\phi(T|a^i)\right] d\zeta - 2c'(|\phi(T|a^j) - .5|) = \frac{1}{2}$$

This evaluates to

$$\frac{1}{8}\left[1 - 2\phi(T|a^j) + 2\phi(T|a^i)\right] + \frac{3}{8}\left[2\phi(T|a^j) + 1 - 2\phi(T|a^i)\right] - 2c'(|\phi(T|a^j) - .5|) = \frac{1}{2}$$

$$\phi(T|a^j) - \phi(T|a^i) = 4c'(|\phi(T|a^j) - .5|) \qquad (**)$$

A similar derivation shows the other first order condition is

$$\phi(C|a^i) - \phi(C|a^j) = 4c'(|\phi(C|a^i) - .5|). \qquad (***)$$

The left hand sides of these two expressions are always equal. This implies that $|\phi(C|a^i) - .5| = |.5 - \phi(C|a^j)|$. It follows that $\phi(C|a^j) = 1 - \phi(C|a^i)$. With this, the first order conditions are rewritten,

$$\phi(T|a^j) = \frac{1}{2} + 2c'(|\phi(T|a^j) - .5|)$$

$$\phi(C|a^i) = \frac{1}{2} + 2c'(|\phi(C|a^i) - .5|)$$

Notice $\phi(T|a^j) = \phi(C|a^i) = .5$ satisfies the first order conditions. To check when these are welfare maximizing, consider the second order condition. The first derivative is:

$$\frac{\partial W}{\partial \phi(T|a^j)} = \int_0^{.5} (2\zeta - 1)Pr(a^j|\zeta) - u_\zeta(a^j)\frac{1}{2} + u_\zeta(a^i)\frac{1}{2}d\zeta + \int_{.5}^{1} (2\zeta - 1)Pr(a^j|\zeta) + u_\zeta(a^j)\frac{1}{2} - u_\zeta(a^i)\frac{1}{2}d\zeta$$

$$-2c'(|\phi(T|a^j) - .5|) = 0.$$

Therefore the second derivative is:

$$\frac{\partial^2 W}{\partial \phi(T|a^j)^2} = \int_0^{.5} (2\zeta - 1)\left(-\frac{1}{2}\right) - (2\zeta - 1)\frac{1}{2}d\zeta + \int_{.5}^{1} (2\zeta - 1)\frac{1}{2} + (2\zeta - 1)\frac{1}{2}d\zeta - 2c''(|\phi(T|a^j) - .5|).$$

$$\frac{\partial^2 W}{\partial \phi(T|a^j)^2} = \int_0^{.5} -2\zeta + 1d\zeta + \int_{.5}^{1} 2\zeta - 1d\zeta - 2c''(|\phi(T|a^j) - .5|).$$

$$\frac{\partial^2 W}{\partial \phi(T|a^j)^2} = -2\frac{1}{8} + .5 + 2\frac{3}{8} - .5 - 2c''(|\phi(T|a^j) - .5|)$$

Therefore the second order condition is:

$$\frac{\partial^2 W}{\partial \phi(T|a^j)^2} = .5 - 2c''(|\phi(T|a^j) - .5|) \leq 0$$

Which implies that $\phi(T|a^j) = \phi(C|a^i) = .5$ maximizes welfare if and only if $c''(0) \geq 1/4$. This is intuitive when checked against equations (**) and (***), and again shows that profit maximizing equilibrium exhibits excessive product differentiation.