Transition Dynamics in Equilibrium Search*

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Abstract

We study a dynamic equilibrium search model where sellers differ in their urgency to liquidate an asset. Buyers strategically make price offers without knowing a given seller's urgency. We study liquidity and price dynamics on the transition path after an unexpected shock. Generically, the transition includes a phase where all buyers offer the same price, causing a market collapse; however, price dispersion resumes in finite time, leading to a recovery where both types make sales. We show that prices and liquidity can overshoot before converging to the steady state. When relaxed sellers randomly become desperate, dampening oscillations can occur.

Keywords: Equilibrium search, transition paths, dynamic price formation, seller composition, motivated sellers, fire sales, liquidity, endogenous reversals

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1 Introduction

Search frictions naturally define the illiquidity of an asset: it takes time for a seller to find a buyer who offers an acceptable price, thereby preventing immediate liquidation. This is especially true after negative market shocks, when more sellers may feel greater urgency to sell. With prices rapidly changing around them, sellers must decide whether to forego current offers in hopes of better opportunities later. Likewise, the evolution of prices and the composition of sellers in the market require buyers to strategically decide when to enter the market and what price to offer, not knowing when they will match with another seller or what price that seller will accept.

In this paper, we analyze liquidity and price dynamics following an unexpected market shock in an equilibrium search framework. In the model, a seller has an asset that is less useful to her than to potential buyers. The asset could provide some benefits while awaiting a buyer, but the usefulness varies idiosyncratically: *relaxed sellers* get more use from the asset than *desperate sellers*, although the types are indistinguishable to buyers. This model could apply in many settings, but to fix ideas, consider a car rental firm when it decides to sell some of its vehicles. The firm could be selling the car simply to update or standardize some of its fleet. Alternatively, the firm may be experiencing financial hardship and needs to liquidate some cars to cover loan obligations. In aggregate, the market will always have some sellers of each type (including individual sellers, who have a similar range of motives). However, a shock could dramatically alter the mixture of sellers; for instance, widespread travel restrictions during a pandemic could force many car rental firms (and rideshare drivers) to quickly sell vehicles.

In our model, the asset's value to interested buyers is commonly known (so adverse selection is not an issue), but a buyer cannot observe the seller's urgency to sell. In making an offer, the buyer knows that a lower price also comes with a lower probability of acceptance. When these forces exactly balance, identical assets can be sold at two different prices, violating the law of one price.[1] Indeed, asset liquidity depends on what fraction of

[1]Diamond (1987) and Albrecht, *et al* (2007) provide the closest model to our steady state analysis, with

buyers offer full price. If more buyers seek to find the asset at a discounted price, some sellers reject these offers and the asset requires more time to sell, making it less liquid.

Equilibrium search models often focus on steady state analysis, where new entrants precisely replace those exiting the market. Although these assumptions provide tractability, such a framework is inadequate for studying short- and medium-term behavior after market disruptions. For example, after a sudden influx of either seller type, buyers can only slowly reach and transact with the excess sellers due to the search friction, preventing an immediate return to steady state. We characterize the unique equilibrium transition path following any unexpected market shocks.

We find that price dispersion is fragile in the short run, but robust in the medium run. That is, the price distribution collapses to a single price in almost every equilibrium transition. Even a small imbalance in the mixture of relaxed and desperate sellers (where the latter face greater urgency to sell) can make it optimal for buyers to exclusively target the relatively-abundant segment of the market. However, price dispersion is restored in finite time, since catering to one group of sellers ensures the relative build up of the other group. Because of this, prices often do not follow a monotonic path toward steady state.

This strategic targeting by buyers also leads to endogenous reversals in asset liquidity and inventory build-up. We illustrate these dynamics after an unanticipated, temporary surge of desperate sellers. Transaction prices drop discretely as buyers exclusively target desperate sellers and lower the price they offer such sellers, drastically reducing asset liquidity. This strategic shift reduces the stock of desperate sellers but unintentionally builds the stock of relaxed sellers, who reject all offers. Eventually, this untapped supply becomes so large that buyers cannot afford to ignore them. At this point, market liquidity experiences a sudden thaw, unleashing a rush of transactions with relaxed sellers. In fact, prices overshoot their steady state level, resembling an overheated market, because full price offers are more abundant while targeting excess relaxed sellers. This non-monotonic price adjustment

price posting in the former and bargaining over prices in the latter. Price dispersion occurs due to differences in valuation here; other sources of dispersion include differences in search costs (Salop and Stiglitz, 1977; Stahl, 1989), number of simultaneous price quotes (Burdett and Judd, 1983), or time remaining until a deadline (Akın and Platt, 2012).

is exacerbated in the model extension where relaxed sellers may randomly become desperate. There, prices can oscillate while converging to steady state. Price overshooting after market shocks have been documented in housing and related financial assets (Semmler and Bernard, 2012), stocks (Zeira, 1999), and commodities (Browne and Cronin, 2010; Baek and Miljkovic, 2018). Allen and Gale (2004) and Barlevy (2012) also document that asset prices overshoot when they crash in a fire sale.

Other applications of the model include housing, labor, over-the-counter financial assets, and money search. Homeowners can vary in their urgency, with relaxed sellers needing more space for a growing family while desperate sellers might be relocating to another city. The collapse of a local industry (such as auto makers in Detroit) would disrupt this housing market, adding many desperate sellers. Turning to labor markets, workers who are otherwise identical could have different reservation wages based on accumulated savings, but a drop in the stock market could reduce those savings, making more workers desperate. The motives of a financial-asset seller are similarly opaque, with some seeking to rebalance their portfolio while others are in need of quick cash. Finally, in the money search environment, our model could add nuance to the medium-of-exchange function of money, since some producers might have waited longer to trade and thus be more desperate, allowing for dispersed fiat-money prices in steady state and rich dynamics after a market shock, such as a monetary injection.

1.1 Related Literature

We contribute to an emerging branch of the equilibrium search literature, examining transition dynamics when a market is not in steady state (Duffie, *et al*, 2007; Weill, 2007; Lagos, *et al*, 2011); most of this literature consider a scenario similar to our first experiment, in which the market incurs an unanticipated shock of sellers needing to liquidate their asset. This leads to a drop in the market price followed by a monotonic recovery.

Our results are distinguished in that price dispersion can occur, and that prices and liquidity can overshoot during a transition. These results all stem from buyers randomizing with take-it-or-leave-it offers, rather than setting prices through Nash bargaining with full knowledge of the seller's type. Random offers are a natural consequence of imperfect information, and prevent some efficient trades from occurring. Transition dynamics critically depend on the rejected offers, since more rejections lead to excess accumulation of the rejecting sellers, which ultimately necessitates a reversal in the price and liquidity path.

In the preceding papers and ours, assets are homogenous but individuals receive idiosyncratic value from them. Alternatively, assets could differ in quality that is privately known by the seller; Camargo and Lester (2014) considers dynamic transitions in this adverse selection environment.[2] While their buyers make random offers, a key difference is that they follow a cohort of sellers until they exit, without any continuing inflow of sellers. All buyers offer the same price, monotonically rising over time, which is accepted at first by sellers with low-quality assets, but eventually by all sellers. While liquidity gradually increases, neither price dispersion nor overshooting can occur; the former requires balanced inflows of both types, while the latter requires an excess buildup of sellers in the unserved portion of the market.

Indeed, price dispersion only appears in a few models of dynamic transitions, but with different underlying mechanisms. In the money search model of Burdett, *et al* (2017),[3] price dispersion arises because buyers randomly receive multiple price quotes. Since the probability of multiple quotes is exogenously fixed, dispersion is maintained throughout the transition path. In contrast, our setting indicates that a shock will collapse and later restore the price distribution in almost every transition. Garriga and Hedlund (2020) generate price dispersion through directed search, with liquidity-constrained sellers listing lower prices so as to attract buyers and sell faster. After the shock, list prices fall for those needing a quick sale, but rise for others who are underwater and are prevented from short selling, so list price dispersion increases (the distribution of realized transaction prices is not reported).

[2]Guerrieri and Shimer (2014) and Chang (2018) model this with directed search, where buyers and sellers choose among markets with fixed prices, and consider comparative statics on the steady state rather than dynamic transitions. When average asset quality falls, the market will decrease its price and volume of sales. Chiu and Koeppl (2016) consider dynamics in a similar environment, but an influx of low-quality assets permanently shuts down all transactions which can only be thawed by the government announcing the intention to purchase assets in the future.

[3]For examples of money search model dynamics without price dispersion, see Boldrin, *et al* (1993) and Coles and Wright (1998), as well as the thorough survey in Lagos, *et al* (2017).

Directed search leads buyers and sellers to sort into distinct markets and thereby reveal their type and preserve dispersion throughout the transition. In our model, seller type is private information, exposing buyers to the risk of rejection and potentially leading them to employ a mixed strategy. However, this buyer indifference is temporarily broken after a shock, causing transaction prices and dispersion to collapse for a spell.

The accumulation of excess sellers contributes to propagating a market shock over time; for instance, Guren and McQuade (2013) examine how foreclosures exacerbate downturns in the housing market. Díaz and Jerez (2013) make similar points regarding the housing market, noting that unsuccessful sales in one period add excess sellers to the next period, leading to further propagation of a low price. Outside of a search setting, Kiyotaki and Moore (1997) show that credit frictions also contribute to shock propagation. In their model, a one-time boost in asset productivity pushes up the asset price and thus raises the collateral value. However, liquidity is not a relevant concept in their model, since all assets are repurchased each period. Also, there is no price dispersion in either Díaz and Jerez (2013) or Kiyotaki and Moore (1997), and their transitions follow a monotonic return to steady state in their base model. In our setting, price dispersion contributes to shock propagation by altering the mix of sellers on the market, which drives the eventual reversals of price and liquidity.

Recurring cycles, in which the populations and prices endlessly repeat a given sequence, can occur in some money search models (Boldrin, *et al*, 1993; Coles and Wright, 1998; Burdett, *et al*, 2017) and the consumer search model of Albrecht, *et al* (2013). In the latter, a seller charges a high price, but the buildup of low-valuation customers motivate the seller to offer a periodic sale. The model of Maurin (2020) produces a similar cycle driven instead by adverse selection, where only low-quality assets trade for several periods, building up high-quality assets until they are sufficiently plentiful. Then buyers offer a high price, buying up all assets in a single period and reseting the cycle. In both models, uninformed agents make a take-it-or-leave-it offer (as in our setting) but there is a single price per period.[4] Only cycle dynamics are considered, not transitions to steady state. Liquidity is constant (and low) for most of the cycle, only occasionally spiking to clear the market. In a similar vein, our model dynamics depend on which sellers are targeted at each phase of the transition, but this allows liquidity to freeze up (dropping below steady state), overheat (jump above steady state), and gradually cool down.[5]

These equilibrium dynamics enrich the theory of fire sales that investigate the forced sale of an asset at a dislocated price below its fundamental value (Schleifer and Vishny, 1992, 2011).[6] Assets sold in haste impose an externality on other asset holders, as prices drop industry-wide. Hence, the concept of a fire sale readily suggests the need for a search model, since it posits that better price offers exist (now or in the future) but are not readily encountered. Yet fire sales are also temporary phenomenon, not permanent conditions. The illiquidity and loss of dispersion in our transition path is consistent with the temporary freezing of a portion of the market after a negative shock, such as the dramatic decline and eventual rebound in non-foreclosure home sales during the Great Recession (Florida Realtors, 2019). Even when buyers are generally aware that an an industry, firm, or individual is distressed, there is still uncertainty as to the degree of distress and thus how desperate they are to unload assets; our model is relevant to this imperfectly-known desperation.

[4]A mixed strategy steady state can exist in Maurin (2020), but it is not unique and is unstable, which explains why cycles are possible. Our mixed strategy steady state is unique and globally stable. In addition to differences in asset quality, sellers in Maurin (2020) have idiosyncratic differences in their value of the asset similar to ours. However, if asset quality were homogenous in their model, all dynamics evaporate, with high value traders buying from low value traders at the low value.

[5]This cannot occur in an endless cycle because continuity of the Bellman equations results in continuous prices over time. A desperate phase cannot jump to a relaxed phase, for instance, because prices are continuously falling in the former and continuously rising in the latter. This pushes either phase toward the dispersed path, and make it unprofitable to offer only one price thereafter. Even in our extended model presented in Section 5.3, the oscillations dampen toward the steady state rather than sustaining a true cycle. There, we discuss the extended model of Kiyotaki and Moore (1997) with user-specific improvements to the asset, which shows similar dampening oscillations.

[6]For example, when a bank or homeowner liquidates an asset to cover short-term borrowing costs, Campbell, *et al* (2011) report that the asset sells at a 27% discount relative to market value. When airlines sell airplanes to pay back loans, they are discounted 10-20% according to Pulvino (1998).

2 Baseline Model

A homogenous asset produces instantaneous value y in a continuous time environment. However, an idiosyncratic shock makes the asset less useful for their current owner, permanently dropping the instantaneous value to x for *relaxed* owners, or dropping to $x - c$ for *desperate* owners. Importantly, the shock severity is private information that the owner will not want to disclose, as we will show.

When this exogenous shock occurs, the owner is interested in liquidating the asset and enters the market as a seller. Any buyer in the market will once again obtain value $y > x$ from the asset, so any sale will be efficient. Sellers encounter homogeneous potential buyers at exogenous rate λ,[7] whereupon the buyer makes a take-it-or-leave-it offer; if rejected, both parties continue their search, with no recall of past offers. Time is discounted at rate ρ.

Relaxed sellers are assumed to enter the market at exogenous Poisson rate η, and at rate δ for desperate sellers; they exit the market whenever their asset is sold. The measure of relaxed sellers in the market at time t is $h_r(t)$ and desperate sellers is $h_d(t)$. For notational simplicity, we omit the function of time and use Newton's notation for time derivatives.

We assume that $\rho < \lambda$ and $\rho < 1$ to ensure that search yields opportunities with enough frequency to be worth the time cost.[8]

2.1 Buyers

Upon purchasing the asset, a buyer enjoys instantaneous value y from the good perpetually, with a present discounted value of $\frac{y}{\rho}$. On meeting a seller, the buyer must decide whether to offer price p_d, which only desperate sellers are willing to accept, or price p_r, which any seller will accept. We refer to $p_r - p_d$ as the *desperation discount*. Since he cannot distinguish

[7] In the baseline model, we take the search friction to be constant, even throughout a dynamic transition. This would mean that sellers always find buyers at the same rate, although the distribution of offers from those buyers may vary. In Section 5.1, we endogenize the meeting rate λ based on the population of sellers and free entry on buyers. This generates a constant meeting rate if the short side of the market always matches (see footnote 22). Even with a Cobb-Douglas matching function, the match rate is constant in relaxed and dispersed phases, and qualitatively similar in desperate phases.

[8] Both requirements are used in Lemma 3, which appears in the proof of Proposition 3 to ensure that, in the desperate region, buyers are willing to exclusively target desperate sellers.

seller types, he faces the tradeoff that the p_d offer will cost him less but is less likely to be accepted. Specifically, the *desperation ratio* is the fraction of sellers willing to accept the lower price, denoted:

$$\phi \equiv \frac{h_d}{h_r + h_d}. \tag{1}$$

The relative gain to a buyer from targeting desperate sellers (by offering p_d) is:

$$\Pi \equiv \phi \left(\frac{y}{\rho} - p_d \right) - \frac{y}{\rho} + p_r, \tag{2}$$

since the offer p_r is accepted for sure, but the p_d offer is only accepted with probability ϕ. If indifferent, meaning $\Pi = 0$, the buyer can employ a mixed strategy[9] offering p_d with probability μ. Individual rationality requires that $\mu = 1$ if $\Pi > 0$ and $\mu = 0$ if $\Pi < 0$.

We note that in this baseline model, buyers do not make inter-temporal decisions; but this is merely for analytic convenience. In Section 5.1, we extend the model to allow strategic entry and exit of buyers over time, making the match rate λ endogenous.

2.2 Sellers

The asset provides income to the seller until it is sold: relaxed sellers collect x per unit of time,[10] while desperate sellers collect $x - c$, where $c > 0$. We assume that $y \geq x$, so that buyers value the asset more than any seller, making all transactions efficient. Indeed, if $y < x$, then no buyer would offer a price that relaxed sellers would accept, effectively excluding relaxed sellers from the market.

Let V_d denote the present value of expected utility for a desperate seller:

$$\rho V_d = x - c + \dot{V}_d + \lambda \left[\mu \left(p_d - V_d \right) + (1 - \mu) \left(p_r - V_d \right) \right]. \tag{3}$$

[9] This can be interpreted as randomization by the individual buyer, or as a fraction of the buyer population who always offers the desperate price.

[10] Recall that the asset is homogenous from the perspective of the buyers; so any difference in the asset value is idiosyncratic to the seller's needs (such as selling a home to move to a new city, or needing liquidity for other purchases), rather than the asset's fundamental productivity.

These desperate sellers earn $x - c$ from the asset each unit of time, and encounter buyers at rate λ. The $\dot{V}_d$ term captures any change in the future value of search, anticipating changes in μ, p_d, or p_r. Note that desperate sellers are willing to accept either price. The equilibrium desperate price makes the desperate seller indifferent between accepting the offer or continuing her search:

$$p_d = V_d. \tag{4}$$

The present value of expected utility for a relaxed seller, V_r, is:

$$\rho V_r = x + \dot{V}_r + \lambda(1 - \mu)\left(p_r - V_r\right). \tag{5}$$

Relaxed sellers earn x from the asset each unit of time, and while they encounter buyers at the same rate λ, they reject all desperate offers. The equilibrium price will push the relaxed sellers to indifference:

$$p_r = V_r. \tag{6}$$

The key difference between Eqs. 3 and 5 is that relaxed sellers avoid the cost c, which ensures that $V_r > V_d$. Both are free to accept either offer, but since $p_r > p_d$, it is always optimal for relaxed sellers to reject desperate price offers.

2.3 Population Dynamics

The fraction of desperate sellers in the market is particularly important in this model, since this becomes the probability with which p_d offers are accepted. We thus track how the stock of sellers in the market adjusts over time. First, consider the population of desperate sellers. These enter the market at rate δ, but they accept any offer, and thus exit the market at rate λh_d. Thus, the net change in their population is:

$$\dot{h}_d = \delta - \lambda h_d. \tag{7}$$

Relaxed sellers, on the other hand, enter the market at rate η, but only accept relaxed price offers, exiting at rate $\lambda(1-\mu)$. Thus:

$$\dot{h}_r = \eta - \lambda(1-\mu)h_r. \tag{8}$$

2.4 Equilibrium Definition

For a given initial seller population $h_r(0)$ and $h_d(0)$, a search equilibrium consists of price functions p_r and p_d, seller expected utility functions V_r and V_d, population functions h_r and h_d, and buyer strategy function μ at each point in time such that:

1. Seller utility correctly anticipates market conditions (Eqs. 3 and 5).

2. Prices are optimally set for their respective targets (Eqs. 4 and 6).

3. Seller populations obey the law of motion (Eqs. 7 and 8).

4. Buyers use optimal pricing strategies: $\mu = 0$ if $\Pi < 0$ and $\mu = 1$ if $\Pi > 0$.

The fourth requirement ensures that buyers offer the price with the highest expected payoff. If $\Pi = 0$, then any $\mu \in [0, 1]$ is admissible, though the other equilibrium requirements will pin down its value. If offering either price is equally profitable, $\Pi = 0$ rearranges to imply a specific relationship between the two prices:

$$p_r = \phi p_d + \frac{y(1-\phi)}{\rho} \text{ if } \mu \in (0, 1). \tag{9}$$

2.5 Steady State Equilibrium

We now solve these equilibrium conditions in steady state, finding initial conditions $h_r(0)$ and $h_d(0)$ such that $\dot{h}_r = \dot{h}_d = 0$ for all t. Two possibilities exist, depending on parameter values. In the *relaxed steady state* equilibrium, only the relaxed sellers are targeted because there are too few desperate sellers to be worth offering p_d. In the *dispersed steady state* equilibrium, both prices are offered.[11]

[11] A third scenario where only desperate prices are offered is not possible in steady state, because relaxed sellers would never exit, causing h_r to grow without bound.

Table 1: Steady State Solution

	Dispersed	Relaxed
	occurs when	
	$\beta\delta > \eta$	$\beta\delta \leq \eta$
p_r	$\frac{x}{\rho}$	
p_d	$\frac{x-c}{\rho} + \frac{\eta\lambda(y-x)}{\delta\rho^2}$	$\frac{x-\beta(y-x)}{\rho}$
h_r	$\frac{\beta\delta(\rho+\lambda)}{\rho\lambda} - \frac{\eta}{\rho}$	$\frac{\eta}{\lambda}$
h_d	$\frac{\delta}{\lambda}$	
μ	$1 - \frac{\rho\eta}{(\rho+\lambda)(\beta\delta-\eta)+\rho\eta}$	0

The steady state is always unique, as shown in the following proposition. For notational brevity, we define:

$$\beta \equiv \frac{\rho}{\rho+\lambda} \cdot \frac{c}{y-x}. \tag{10}$$

This measure of discounted delay costs determines whether price dispersion occurs in steady state. The dispersed steady state occurs when $\beta\delta > \eta$, so that $\mu \in (0, 1)$. Moreover, the relaxed steady state occurs if $\beta\delta \leq \eta$: buyers do not target desperate sellers if there are relatively few of them entering, they are fairly patient, or their cost of delay is small.[12]

Proposition 1. *The unique steady state solution is depicted in Table 1.*

Uniqueness occurs because an individual buyer finds it less profitable to target desperate sellers as other buyers target them more heavily (μ increases). Such targeting will decrease the desperation ratio, increasing the probability that the buyer is matched with a relaxed seller. The desperation discount also increases with μ, but this effect is always dominated by the desperation ratio effect. This uniqueness suggests a stable system, where deviations from the steady state will automatically tend back to the steady state, although that is a claim which precisely calls for the dynamic solution we characterize in Section 3.

[12] This can be seen by inserting the equilibrium prices and populations into Eq. 2.

This market faces a single search friction, λ, that generates two important departures from perfect competition. Most obviously, transactions do not occur instantaneously; rather, the asset will spend some amount of time on the market before an interested buyer makes an acceptable offer. This explains why there are positive stocks of both seller types awaiting a buyer. These stocks shrink as meetings become more frequent, but only disappear in the competitive limit ($\lambda = +\infty$) where new sellers can immediately find a buyer.

A more subtle effect is that the desperation discount relies on slow transactions, moving inversely with λ. Desperate sellers only accept lower prices because of the additional costs they incur while awaiting the next offer. Thus, the search friction is also responsible for the violation of the law of one price. However, price dispersion requires λ to be below a finite threshold; as the search friction is eased beyond that threshold, the desperation discount becomes too small for buyers to profitably offer the low price. Thus, reducing search frictions will first be visible in collapsing the steady state price distribution to only offer p_r; but then h_r and h_d will shrink toward 0 in the limit.

Corollary 1. *The desperate price p_d is increasing in λ, approaching p_r as $\lambda \to \infty$. Meanwhile β, h_r, and h_d are decreasing in λ, approaching 0 in the limit. Thus, a dispersed steady state equilibrium only occurs if $\lambda < \frac{\rho c \delta}{(y-x)\eta} - \rho$.*

Finally, we emphasize that only desperate sellers earn informational rents in this setting. When a relaxed seller accepts an offer p_r, he is exactly compensated for the flow of utility x that he gives up; indeed, further search will at best produce the same offer. In contrast, when a desperate seller accepts p_d, he requires compensation for not only for the flow value of the asset, $\frac{x-c}{\rho}$, but also the expected future benefit of search, in which he could inadvertently be offered p_r. An important practical consequence is that a desperate seller would never prefer to reveal his desperation. If he did, the buyer would make $\frac{x-c}{\rho}$ as a take-it-or-leave-it offer, leaving him strictly worse off.[13]

[13] By revealing his desperation, a seller could potentially increase his matching rate (which is exogenous in our base model); however, the same information that would draw more buyers would also be used to offer $\frac{x-c}{\rho}$. This leaves the desperate seller with zero surplus from the match, making the rate at which it occurs irrelevant. Directed-search models sidestep this issue by committing each submarket to a specified price.

Corollary 2. *The desperate price p_d is always strictly greater than $\frac{x-c}{\rho}$; hence, desperate sellers strictly benefit from not disclosing their private information.*

We find the same result in an augmented model with a third seller type, with the same level of desperation $x - c$ but whose status is public information. In equilibrium, three prices can emerge. Buyers always offer $\frac{x-c}{\rho}$ when paired with a known desperate seller, who accepts. Otherwise, buyers can randomize price offers when paired with sellers whose status is unknown, as in our base model. Undisclosed desperate sellers would be strictly worse off by signaling their status and thus inviting an offer at the known-desperate-seller's price.

3 Dynamic Transition Characterization

We now fully characterize the equilibrium transitions. When the initial population of relaxed sellers $h_r(0)$ and/or desperate sellers $h_d(0)$ are not at their steady state levels, we can derive their unique transition paths. It is important to note that this is a deterministic transition, with a future path that is commonly known by all buyers and sellers. Absent any frictions, the potential for arbitrage would force an immediate transition to the long run outcome. Despite knowing how prices will adjust over time, buyers and sellers in our model only occasionally have opportunities to interact; thus, their populations only adjust slowly towards the long run.

The dynamic path can be characterized as passing through phases. When a shock to the economy causes an imbalance in populations of seller types, buyers will typically target the relatively plentiful group, generating either a *desperate* or *relaxed* phase. In a *dispersed* phase, buyers target both types of sellers.

3.1 Possible Dynamic Paths

Lemma 1 and the accompanying Table 2 establish the solution for each phase. We let q_d denote the difference between the initial population of desperate sellers and $\frac{\delta}{\lambda}$ (its steady state level), while q_r denotes the difference between the initial population of relaxed sellers

Table 2: Dynamic Paths

	Desperate Phase	Dispersed Phase	Relaxed Phase
	occurs when		
	$\Pi(t) > 0$	$\Pi(t) = 0$ and $\mu(t) \in (0,1)$	$\Pi(t) < 0$
$p_r(t)$	$\frac{x}{\rho}$		
$p_d(t)$	$\frac{x-c}{\rho} + a_d e^{\rho t}$	$\frac{x}{\rho} - \frac{y-x}{\rho} \cdot \frac{h_r(t)}{h_d(t)}$	$\frac{x-\beta(y-x)}{\rho} + a_r e^{t(\rho+\lambda)}$
$h_r(t)$	$\frac{\eta}{\lambda} + q_r + \eta t$	$\Delta(h_d(t))$	$\frac{\eta}{\lambda} + q_r e^{-\lambda t}$
$h_d(t)$	$\frac{\delta}{\lambda} + q_d e^{-\lambda t}$		
$\mu(t)$	1	$1 - \frac{\eta - h_r'(t)}{h_r(t)}$	0

and $\frac{\eta}{\lambda}$ (its level in the relaxed steady state). We also define a buyer indifference condition $\Delta(h_d)$ as the function:

$$\Delta(h_d) \equiv \beta h_d + \frac{\beta\delta - \eta}{2\lambda} \int_0^1 \frac{(1-s)^{\frac{\rho-2\lambda}{2\lambda}}}{\left(1 - s + \frac{\delta}{\lambda h_d} s\right)^{\frac{1}{2}}} ds. \tag{11}$$

For any desperate population h_d, Eq. 11 specifies a relaxed population $h_r = \Delta(h_d)$ such that offering either reservation price is equally profitable for buyers. As we shall see below, this is applicable when parameters lead to a dispersed steady state. The integral in Eq. 11 accounts for the anticipated evolution of the price distribution (which affects current reservation prices). Contrast this to an environment with a relaxed steady state: since future prices are concentrated entirely on p_r, there is no price further evolution; hence, the integral in Eq. 11 disappears, and equal profits occur when $h_r = \beta h_d$.

Lemma 1. *Given initial seller populations $h_d(0) = \frac{\delta}{\lambda} + q_d$ and $h_r(0) = \frac{\eta}{\lambda} + q_r$, a dynamic equilibrium path is characterized by one of the three phases listed in Table 2, for some constant a_r or a_d.*

In any phase, the relaxed price is constant at $\frac{x}{\rho}$. Intuitively, the relaxed sellers have no reason to accept a lower price, since this exactly replaces their current utility; and buyers have no reason to make a larger take-it-or-leave it offer, since everyone is willing to accept that price.

Turning to desperate population dynamics, note that the same transition path applies in any phase. This holds because desperate sellers accept every offer that is made in equilibrium and thus exit the market at a constant rate λ. Also, the population monotonically approaches its steady state level over time, or stays at that level for all t if $q_d = 0$.

On the other hand, relaxed population dynamics depend on the phase. In a relaxed phase, the relaxed population monotonically approaches its steady state level. However, in a desperate phase, relaxed sellers reject all offers and thus their population increases over time. In a dispersed phase, buyers have to be indifferent between making low or high price offers. This can only occur if the relaxed population has a specific relationship with the desperate population, as indicated by $h_r(t) = \Delta(h_d(t))$. Indeed, the set of (h_d, h_r) pairs that can be part of a dispersed transition path form a one-to-one mapping, with h_r increasing in h_d.[14]

Finally, the solution for the desperate price is unique up to a constant — a_d for the desperate phase or a_r for the relaxed phase — which are pinned down in the next subsection by transitions between phases. Here we note that the desperate price is increasing over time if and only if this constant is positive.

In the next subsections, we consider how these phases interact to form a full dynamic path. This behavior differs depending on whether the unique steady state has degenerate or dispersed prices.

3.2 Transition toward a Relaxed Steady State

First, suppose that parameters are such that only the relaxed price will be offered in steady state. We consider the dynamic transition starting from any initial population. Figure 1.A divides the space of seller populations into two regions, depending on which phase occurs

[14] Since $h_d(t) \to \frac{\delta}{\lambda}$ as $t \to \infty$, the integral in $\Delta(h_d)$ will approach $\frac{2\lambda}{\rho}$, and thus $h_r(t)$ approaches its dispersed steady state value.

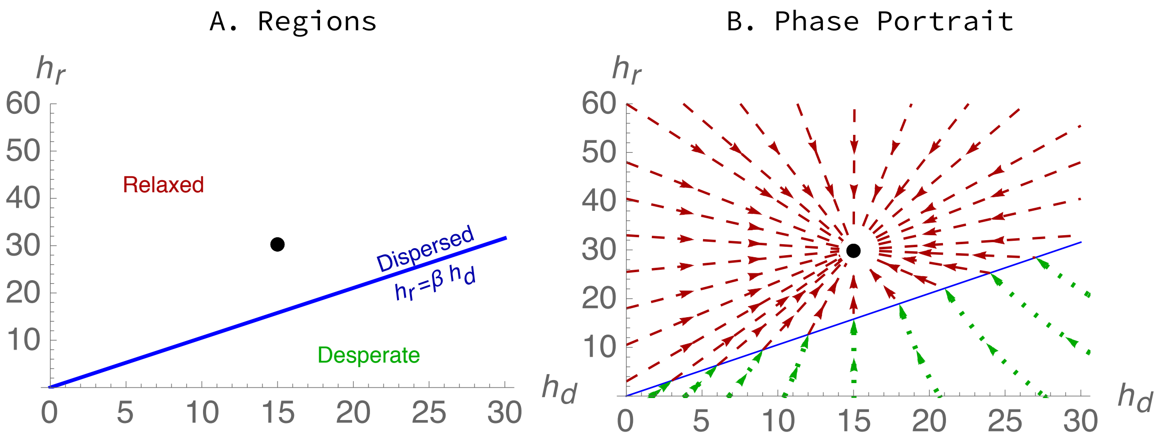

Figure 1: **Transitions to a Relaxed Steady State:** Panel A divides the population state space (h_d, h_r) into regions based on the phase which occurs there. Panel B indicates the path of seller populations from any initial state, with dashed arrows in a relaxed phase and dotted arrows in a desperate phase. The solid line indicates the boundary between these phases. The solid dot marks the relaxed steady state populations. ($y = 1.0285$, $x = 1$, $c = 1$, $\eta = 10$, $\delta = 5$, $\rho = 0.01$, and $\lambda = 0.33$)

in that region. Figure 1.B illustrates the transition paths in a phase portrait (derived in Proposition 2). In both figures, steady state populations are indicated by the dot. When the path is dashed in Figure 1.B, it indicates that buyers only offer relaxed prices; when it is dotted, buyers only offer desperate prices. Buyers are only indifferent between offering the two prices along the solid line.

In the relaxed region (where $h_r > \beta h_d$), buyers only offer the relaxed price due to the relative abundance of relaxed sellers. For instance, starting from $h_d = 0$ and $h_r = 10$, both populations are below their steady state levels (15 and 30, respectively); therefore, new entrants (δ or η) outpace exiting sellers (λh_d or λh_r). Every seller accepts every offer, so both populations rise on a straight trajectory toward steady state. We note that prices p_r and p_d stay at their steady state values throughout this region.

In the desperate region (where $h_r < \beta h_d$), relaxed sellers are relatively scarce, so buyers find it optimal to initially target desperate sellers. The population of desperate sellers may rise or fall, depending on whether they are below or above the steady state, respectively; but the population of relaxed sellers steadily climbs as they enter the market but accept no offers. This ensures that eventually buyers will shift to target relaxed sellers. For instance, starting from $h_d = 27.5$ and $h_r = 0$, both populations follow the desperate path until $h_d = 21$ and $h_r = 21.5$, after which they continue on a relaxed path.

Significantly, the dynamic transition can have at most a momentary dispersed phase where both prices are offered, as shown in Proposition 2 below. In the interior of either region, buyers would be strictly worse off if they targeted the other seller. On the boundary between these regions (where $h_r = \beta h_d$), offering either price is equally profitable for an instant. Yet a dispersed equilibrium would have relaxed sellers still rejecting some offers, so h_r would grow faster than h_d, making p_d offers unprofitable thereafter. Note that the desperate price p_d is the same throughout the relaxed region and at any point on the dispersed region, since all future offers will be of the constant relaxed price.

The interesting price dynamics occur in the discrete shift from desperate to relaxed phases. The realized transaction price (p_d) starts low and gradually rises over time, hitting its steady state value at the switch to the relaxed phase. However, at that instant, the realized transaction price jumps, as buyers switch to only offer p_r. This happens when desperate sellers are disproportionately abundant ($h_r < \beta h_d$).

Proposition 2 establishes that this behavior holds generally.

Proposition 2. *Suppose that $\beta\delta \leq \eta$, so that the relaxed steady state exists. Given initial populations $h_d(0)$ and $h_r(0)$:*

- ***Case 1:*** *If $h_r(0) \geq \beta h_d(0)$, then for all t the dynamic transition must follow the relaxed phase with $a_r = 0$ until reaching steady state.*

- ***Case 2:*** *If $h_r(0) < \beta h_d(0)$, then for $t < T$ the dynamic transition must follow the desperate phase with $a_d = \frac{\lambda c}{\rho(\rho+\lambda)} e^{-\rho T}$, where T satisfies $h_r(T) = \beta h_d(T)$. For $t \geq T$, it must follow the relaxed phase with $a_r = 0$ until reaching steady state.*

Recall that, as the search friction declines (λ rises), it eventually crosses a threshold where only the relaxed steady state can occur. In a low-friction market, there is no price dispersion during transition or steady state; rather, at any given time, only p_d or (eventually) p_r is offered.

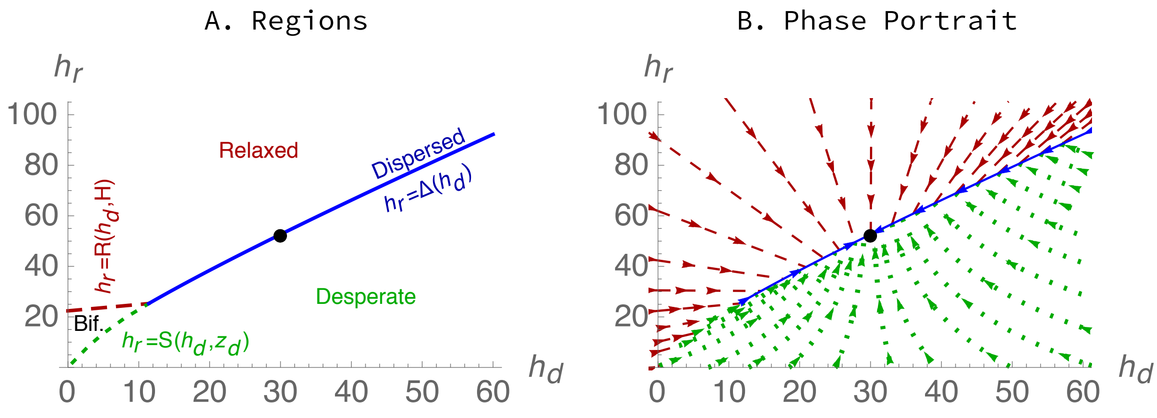

Figure 2: **Transitions to a Dispersed Steady State:** Panel A divides the population state space (h_d, h_r) into regions based on the phase which occurs there. Panel B indicates the path of seller populations from any initial state. The solid dot marks the dispersed steady state populations. ($y = 1.0285$, $x = 1$, $c = 1$, $\eta = 10$, $\delta = 10$, $\rho = 0.01$, and $\lambda = 0.33$)

3.3 Transition toward a Dispersed Steady State

Next, suppose that parameters would lead to both prices being offered in steady state. This produces richer behavior in the transition paths. For a narrow set of initial seller populations (of measure zero), the transition always remains in a dispersed phase. Generically, however, the path will start on a relaxed or desperate phase, then eventually reach the dispersed phase (which forms the boundary between the two phases as illustrated in Panel B of Figure 2). It is even possible to begin in a relaxed phase, transition to a desperate phase, then conclude in a dispersed phase — which we refer to as a *bifurcated* path.

Thus, the space of possible initial conditions (for h_d and h_r) is partitioned into four regions, illustrated in Panel A of Figure 2, each yielding a unique path to steady state. Specific transition paths are exemplified in Panel B of Figure 2. These regions and phase portrait are representative of the general behavior, which we formally show in Proposition 3 at the end of this section.

In the dispersed region (the solid line in Figure 2.A and B), both prices are offered throughout the dynamic transition, and seller populations approach steady state by following the solid line. For example, at $h_d = 15$ and $h_r = 32$, the dispersed phase will gradually approach the steady state of $h_d = 30$ and $h_r = 52$. Indeed, this region is defined by the function $h_r = \Delta(h_d)$, which ensures that buyers are indifferent between the two offers. Moreover, by making the equilibrium offer μ^*, the populations will adjust so as to move along this path, thereby remaining indifferent.

In the space of all possible initial population levels, this dispersed region has measure zero, indicating that price dispersion almost never continues immediately following a random population shock. Even so, the other regions eventually feed into this dispersed path; thus, price dispersion is later observed in some part of almost every transition path. While not depicted in the figure, the desperate price p_d is still adjusting along this path, rising if h_d is below its steady state level of $\frac{\delta}{\lambda}$, and falling otherwise. Meanwhile, the fraction of desperate offers μ moves in the opposite direction.

In the relaxed region (above the dashed and solid lines in Figure 2.A), relaxed sellers are more plentiful, so buyers exclusively offer p_r initially. This eventually draws down the relaxed population sufficiently to reach the dispersed path (dashed lines reaching the solid line in Figure 2.B), at which point it becomes profitable to offer both prices. For example, starting from $h_d = 0$ and $h_r = 32$, the relaxed phase will build up the desperate population to $h_d = 15$ while holding the relaxed population steady at $h_r = 32$, whereupon the preceding dispersed phase begins. Indeed, it is convenient to describe a relaxed path relative to the $\hat{h}_d$ where it intersects the dispersed path. Any pair $(h_d, R(h_d, \hat{h}_d))$ lies on a relaxed path leading to $\left(\hat{h}_d, \Delta(\hat{h}_d)\right)$, where:

$$R(h_d, \hat{h}_d) \equiv \frac{(h_d - \hat{h}_d)\eta + (\delta - \lambda h_d)\Delta(\hat{h}_d)}{\delta - \lambda \hat{h}_d}, \tag{12}$$

as we will derive in the proof of Proposition 3 below. We also note that $a_r < 0$, meaning the price desperate sellers are willing to accept falls until reaching the dispersed path.

In the desperate region (below the dotted and solid lines in Figure 2.A), relaxed sellers are relatively scarce. Thus, buyers find it profitable to exclusively offer p_d initially. This ensures that the population of relaxed sellers steadily grows until eventually reaching the dispersed path (dotted lines reaching the solid line in Figure 2.B). For example, starting from $h_d = 60$ and $h_r = 52$, the desperate phase will decrease the desperate population to $h_d = 45$ while building the relaxed population to $h_r = 73$; whereupon the dispersed phase begins. We similarly depict any pair $(h_d, S(h_d, \hat{h}_d))$ as part of a desperate path leading to $\left(\hat{h}_d, \Delta(\hat{h}_d)\right)$, where:

$$S(h_d, \hat{h}_d) \equiv \Delta(\hat{h}_d) - \frac{\eta}{\lambda} \ln \frac{\delta - \lambda h_d}{\delta - \lambda \hat{h}_d}. \tag{13}$$

In this region, $a_d > 0$, which implies that desperate sellers insist on higher prices over time until reaching the dispersed path.

Finally, the bifurcated region (between the dashed and dotted lines in Figure 2.A) has very few desperate sellers and relatively more relaxed sellers. For example, consider $h_d = 0$ and $h_r = 15$. This leads buyers to initially target the relaxed sellers, but both populations build until $h_d = 9$ and $h_r = 19$. Momentarily, buyers are indifferent between the two offers. However, the market cannot sustain indifference (like it would moving along the dispersed path); doing so would require the relaxed sellers to build up faster than possible (*i.e.* $h_r' > \eta$, necessitating an impossible $\mu > 1$). Instead, desperate sellers become a strictly more attractive target for a time (dashed lines lead to dotted lines in Figure 2.B). This leads to a remarkable discrete drop in the realized price offers, followed by a steady recovery in a desperate phase. Eventually, both populations build up sufficiently that when buyers are again indifferent (joining the dispersed phase at $h_d = 15$ and $h_r = 32$), the market maintains that indifference with a mixed strategy $\mu < 1$.

For this reason, the dispersed path in Figure 2.A does not extend below $h_d = 12$ (the intersection of the regions). To formally define this beginning of the dispersed path, let $H > 0$ solve:

$$\Delta(H) = \frac{\eta(y - x) + \rho c H}{(y - x)(\delta + H(\rho - \lambda))} H. \tag{14}$$

This solution H indicates the lowest desperate population for the dispersed phase, which also has the highest fraction of desperate offers ($\mu = 1$).

We are then able to characterize the unique equilibrium path for any initial condition as follows. In the proposition, recall that $\hat{h}_d$ indicates the population of desperate sellers where the relaxed or desperate path intersects the dispersed path. In the bifurcated case, $\tilde{h}_d$ indicates where the relaxed path hits $\Pi = 0$ and thus transitions to a desperate path.

Proposition 3. *Suppose that $\beta > \eta/\delta$, so that the dispersed steady state exists, and that initial populations are $h_d(0) = \frac{\delta}{\lambda} + q_d$ and $h_r(0) = \frac{\eta}{\lambda} + q_r$.*

- ***Case 1 (Dispersed):*** *If $h_r(0) = \Delta(h_d(0))$ and $h_d(0) \geq H$, then the transition follows the dispersed path until reaching steady state.*

- ***Case 2 (Relaxed):*** *Let $\hat{h}_d$ be the solution to $h_r(0) = R\left(h_d(0), \hat{h}_d\right)$. If $h_r(0) > \Delta(h_d(0))$ and $\hat{h}_d \geq H$, then the transition follows the relaxed phase while $t < T_r \equiv \frac{1}{\lambda} \ln \frac{\lambda q_d}{\lambda \hat{h}_d - \delta}$, where $a_r = \frac{(y-x)(\beta \hat{h}_d - \Delta(\hat{h}_d))}{\rho \hat{h}_d} e^{-(\rho+\lambda)T_r}$. For $t \geq T_r$, it follows the dispersed path until reaching steady state.*

- ***Case 3 (Desperate):*** *Let $\hat{h}_d \geq H$ be the solution to $h_r(0) = S\left(h_d(0), \hat{h}_d\right)$. If $h_r(0) < \Delta(h_d(0))$ and $h_r(0) < \frac{c - \rho a_d}{y - x} h_d(0)$, then the transition must follow the desperate phase while $t < T_d \equiv \frac{1}{\lambda} \ln \frac{\lambda q_d}{\lambda \hat{h}_d - \delta}$ with $a_d = \frac{c \hat{h}_d - (y-x)\Delta(\hat{h}_d)}{\rho \hat{h}_d} e^{-\rho T_d}$. For $t \geq T_d$, it must follow the dispersed path until reaching steady state.*

- ***Case 4 (Bifurcated):*** *Otherwise, let $\hat{h}_d \geq H$ and $\tilde{h}_d$ be the solutions to $S\left(\tilde{h}_d, \hat{h}_d\right) = \frac{\eta}{\lambda} + \frac{(\lambda \tilde{h}_d - \delta) q_r}{\lambda q_d}$ and $S(\tilde{h}_d, \hat{h}_d) = \frac{c - \rho a_d}{y - x} \tilde{h}_d$. Then the transition must follow the relaxed phase while $t < T_b \equiv \frac{1}{\lambda} \ln \frac{\lambda q_d}{\lambda \tilde{h}_d - \delta}$ with $a_r = a_d e^{-\lambda T_b} - \frac{\lambda c}{\rho(\rho+\lambda)} e^{-(\rho+\lambda)T_b}$. This is followed by a desperate phase while $t < T_d \equiv T_b + \frac{1}{\lambda} \ln \frac{\lambda \tilde{h}_d - \delta}{\lambda \hat{h}_d - \delta}$ with $a_d = \frac{c \hat{h}_d - (y-x)\Delta(\hat{h}_d)}{\rho \hat{h}_d} e^{-\rho T_d}$. For $t \geq T_d$, it must follow the dispersed path until reaching steady state.*

This dynamic evolution provides a more thorough evaluation of the stability of the steady state. In the long run, the market will always return to its steady state populations and prices, rather than perpetually cycling around the steady state. The more surprising aspect is that the transitions bear little resemblance to the eventual steady state. For almost every initial condition, the price distribution collapses. Even when dispersion resumes, the price distribution will not match the steady state offerings except in the limit. The duration of the price distribution collapse (T_d or T_r) does depend on the size of the shock ($|q_d|$).

It is also noteworthy that the dynamic transition path is always unique. This stems from the same source as steady state uniqueness: the desperation ratio serves as a counterweight to μ. This is true even in an initial degenerate phase; buyers view one group as strictly more profitable to target. This shifts the desperation ratio until restoring equal profits, entering the dispersed path.

Note also that the four cases in Proposition 3 are exhaustive, ruling out any other transition patterns. In particular, it is not possible to move from a desperate phase to a relaxed phase. To do so would require further buildup of relaxed sellers beyond the $h_r = \Delta(h_d)$ boundary by excessively targeting desperate sellers. Yet whenever $h_r > \Delta(h_d)$, it becomes unprofitable to offer p_d at all (as shown in Case 2 of the proof), immediately reversing the attempted build up. One can move from relaxed (in the bifurcated region) to desperate because on the $h_r = \Delta(h_d)$ boundary, buyers are not indifferent; all at once, buyers strictly prefer offering p_d, even when everyone else does ($\mu = 1$).

4 Liquidity, Price and Inventory Dynamics

Our model has potential to shed light on transition dynamics of real assets during economic cycles. We investigate the effects of two different shocks. In the first example, we consider a sudden increase in the number of desperate sellers in the market, reflecting a recession when asset prices are likely to fall below steady state. Our model then predicts the equilibrium behavior as the market recovers from this shock.[15] In the second example, we consider a sudden, permanent decrease in the matching rate, evaluating how this search friction affects the equilibrium transition.

As commonly done in the search theoretical literature that analyzes fire sales (e.g. Guerrieri and Shimer, 2014; Maurin, 2020), we illustrate the dynamics in our model economy with a numerical example. We set our seven model parameters as follows. We first normalize

[15] The models of Duffie, *et al* (2007); Weill (2007); Lagos, *et al* (2011); Chiu and Koeppl (2016) each evaluate a similar shock, inserting sellers with high liquidity needs. While these generate a discrete price drop and gradual recovery, only one price is offered (unlike our partial market shutdown) and their recovery is always monotonic.

the entry flow of relaxed sellers to $\eta = 10$, and the utility flow of relaxed sellers to $x = 1$.[16] We set $\rho = 0.01$, so one unit of time could be interpreted as one month. We set the matching rate to $\lambda = 0.33$, so that on average sellers receive offers once a quarter and accept an offer within 4 months in steady state. We set the entry flow rate of desperate sellers as $\delta = 10$, the same as relaxed sellers; however, since desperate sellers accept more offers, only one third of the stock of sellers in the steady-state market are desperate. We set $c = 1$, meaning that desperate sellers receive no utility while they hold the asset. Even so, in steady state, they are only willing to accept a 5% discount relative to the relaxed seller, since they frequently are offered the relaxed price. Finally, the buyer receives utility $y = 1.0285$, modestly more than the utility of a relaxed seller, which motivates buyers to offer p_d 43% of the time.

4.1 Experiment 1: Additional Desperate Sellers

Suppose that the market experiences a sudden influx of additional desperate sellers, keeping the number of relaxed sellers fixed. In Figure 3, we illustrate the dynamic response from a 200% increase in the number of desperate sellers at $t = 0$ (h_d jumps from 30 to 90). Note that each variable in Figure 3 returns to its original steady state value in the limit; but the short run behavior can deviate significantly from the long run outcome.

The shock in this experiment is similar to jumping to the right edge of the phase portrait in Figure 2.B. Thus, the transition begins in a desperate phase, then eventually reaches the dispersed phase. In Panel A of Figure 3, the adjusting population levels are shown as a function of time since the shock. Since desperate sellers are relatively abundant after the shock (ϕ jumps upward in Panel C), buyers exclusively target them initially (μ jumps to 1 from its steady-state level of 0.43 in Panel C). During this desperate phase (which lasts $T_d = 3$ months in the figure), relaxed sellers continue to enter the market but reject all offers; meanwhile, desperate sellers accept offers and exit faster than new entrants replacing them. Together, this reduces ϕ at a fast pace until a critical time ($T_d = 3$) at which some

[16] If η and δ are scaled up proportionally, then the populations h_d and h_r rise by the same proportion but no other equilibrium behavior is altered. Similarly, if x, y and c are raised proportionally, equilibrium prices p_r and p_d will rise proportionally with no other impact.

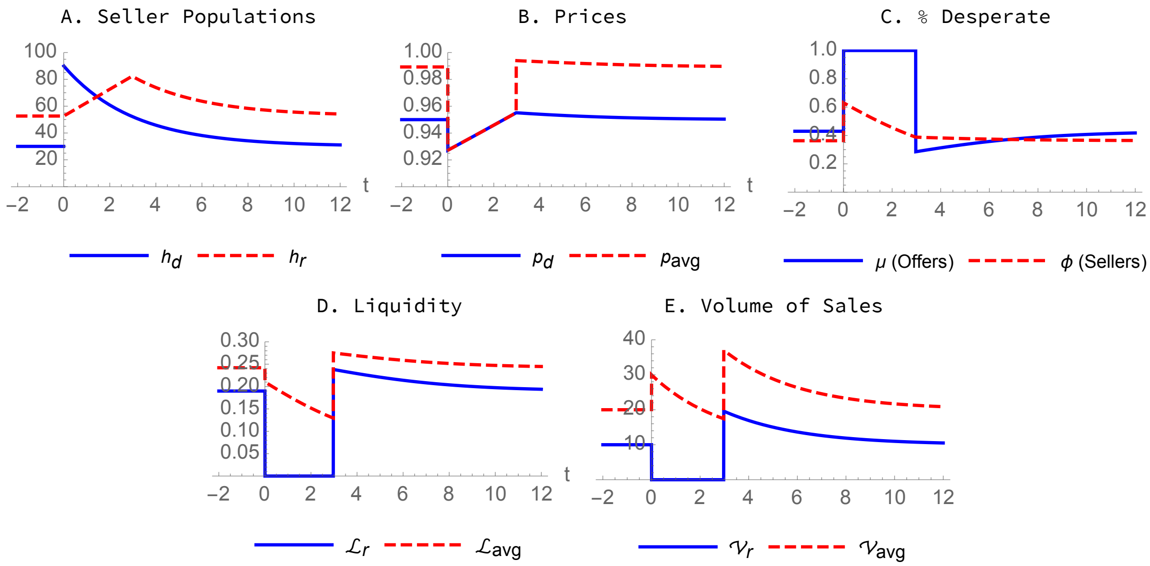

Figure 3: Dynamic response from a 200% increase in the number of desperate sellers at time $t = 0$

buyers resume targeting the relaxed sellers (μ drops to 0.28).

Figure 3 also indicates how the prices respond throughout this transition. The solid line in Panel B depicts a non-monotonic transition in the desperate price, which shadows the fluctuation in future alternatives for the desperate sellers. Following the shock, p_d discretely drops 2.4% because all full price offers have disappeared, making desperate sellers willing to accept a lower price. However, anticipating the return of some full price offers to the market as desperation ratio declines, desperate sellers gradually insist on higher prices (leading to a 3% rise in p_d over the desperate phase) as the critical time approaches. To keep the size of this fluctuation in context, recall that the gain from trade with a relaxed seller is $y - x = 2.85\%$, similar in size to this price fluctuation.

As the dispersed phase begins, buyers offer both prices, but they make more relaxed offers than they eventually will in steady state. This can be seen in Panel C, as the proportion of desperate offers gradually climbs from $\mu = 0.28$ and asymptotically approaches 0.43. This focus on relaxed sellers is not surprising since they are relatively abundant in the market, pushing desperate sellers to require a high p_d to exit search. As the dispersed phase continues, the desperation ratio ϕ falls slightly, causing buyers to reduce the fraction of full-price offers, and making desperate sellers willing to accept lower price offers. The non-monotonic price transition is even more dramatic when measured in terms of realized transaction prices. The dashed line in Panel B indicates the average price at which the asset is sold. During the desperate phase, we see a 6% collapse in asset prices, since all relaxed price offers disappear.[17] When the dispersed phase begins, this decline is reversed by the reintroduction of relaxed offers. Note that price fluctuations are driven by the changes in the desperate (rather than relaxed) price, as well as the frequency of the desperate price, which is consistent with housing price fluctuations documented in Campbell, *et al* (2011).

In Panel D, we show how liquidity $\mathcal{L}$ varies over the transition, measured as the monthly arrival rate of acceptable offers. Note that $1/\mathcal{L}$ provides a measure of inventory, meaning the average number of months required to sell the current stock of listings at the current rate. The solid line indicates liquidity from the perspective of a relaxed seller, with transactions occurring at rate $\lambda(1-\mu)$. Their liquidity completely freezes during the desperate phase, with no sales occurring; but after the dispersed phase begins at time $T_d = 3$, liquidity overshoots, temporarily making it 25% faster to sell than in steady state. Note that desperate sellers have constant liquidity since they accept all offers. The dashed line averages liquidity across all listings, $\lambda\frac{h_d+(1-\mu)h_r}{h_d+h_r}$, *e.g.* the perspective of an uninformed observer of the market. At $t = 0$, relaxed sellers stop transacting, causing a discrete 10% drop in average liquidity. From there, liquidity continues to fall to nearly half its steady state value. This occurs solely due to the composition in the market, as relaxed sellers accumulate (with 0 liquidity) while the desperate sellers are depleted (with λ liquidity). At the critical time of $T_d = 3$, buyers again offer relaxed prices, causing liquidity to rebound 15% above its steady-state level before gradually subsiding.[18]

Finally, we demonstrate how sales volume behaves in Panel E, which is computed as $\lambda(1-\mu)h_r$ for the relaxed sellers, and as $\lambda(h_d + (1-\mu)h_r)$ for the aggregate economy. The

[17] These price dynamics are on a similar scale to the empirical work of Coval and Stafford (2007) studying liquidation of assets by mutual funds. There, in the quarter following a forced liquidation, falling prices reduce the average abnormal stock return by 3%. This rebounds 6.1% over the following four quarters.

[18] Fire sales by mutual funds show similar liquidity dynamics in Barbon, *et al* (2019), though on a shorter time frame. Liquidity suddenly drops at the time of forced sales, remains low for 5 days, then has a similarly sudden rebound.

relaxed sales volume drops to zero during the desperate phase, but rebounds above steady state at $T_d = 3$, due to the accumulated relaxed sellers and more full-price offers being made. Aggregate sales volume spikes at $t = 0$; despite no sales by relaxed sellers, sales opportunities jump then slowly subside with the increased population of desperate sellers. Volume spikes again at T_d as the relaxed sellers resume sales.

In our simulation, the proportion of sellers that are desperate reaches 64% of the market and the average asset price falls 6.5% at the trough. The discount on distressed asset sales reaches as low as 7.4%, whereas inventory reaches 7.7 months. These changes in endogenous variables help us quantify and understand the economic effects of a sudden need for liquidation through a downturn.

To determine how each parameter affects the length of market recovery, we examine their elasticity on the length of the desperate phase T_d. We note that T_d is determined by when the desperate path intersects with the dispersed path, and must be numerically solved due to the non-linearity of the latter. This primarily depends on the slope of the dispersed path in Figure 2, which is to say, the relative rates at which populations h_d and h_r change. If h_d is drawn down slower than h_r, the dispersed path has a steeper slope (rotating around the steady state), so any given desperate path on the right side of Figure 2 will intersect the dispersed path later.

Indeed, this is the only impact of an increase in δ, c, or ρ. Any of these increase β, thereby increasing the slope of $\Delta(h_d)$. These parameters do not affect the desperate path, so the intersection must take more time to reach. Similarly, a decrease in $y - x$ will have the same effect of raising β and hence recovery time; indeed, the value to buyer and seller only matter relative to each other.

A decrease in λ or η will also increase T_d, but here the desperate path also becomes shallower or steeper, respectively. Even so, the direct effect of a steeper $\Delta(h_d)$ dominates in both cases, so that the recovery takes more time. Indeed, evaluated at the preceding parameters, the effect magnitudes (measured as an elasticity) are quite similar, with a 5% change in T_d in response to a 1% change in a given parameter (or 6% in the case of λ).

Lastly, the size of the shock q_d also matters to recovery time, but here a 1% increase in the excess desperate sellers only increases T_d by 0.25%. This is because the extra sellers are added at the beginning of the transition, when the exponential decay of h_d toward steady state is fastest (in terms of absolute levels). In contrast, shocks to parameters move the steady state further away; so effectively, extra sellers must be processed at the end of the desperate phase, after the absolute decline has slowed down.

4.1.1 A Quantitative Easing Intervention

In the preceding experiment, a sudden excess of desperate sellers led to a collapse of asset prices. This might invite an intervention by a central bank, along the lines of the Quantitative Easing (troubled asset purchases by the Federal Reserve during the Great Recession). We investigate the effect of one such intervention. Suppose that the Fed were to purchase assets from distressed sellers, with the intent to hold them until they can be sold at the relaxed price. For simplicity, we assume that they can purchase assets en masse (in an instant), but must sell them subject to the same search frictions as relaxed sellers, and that the Fed will purchase enough to return desperate sellers to their steady state level. We also assume that the intervention was unanticipated, exogenously imposing that this intervention occurs at time $t = 1$ to reflect lags in reacting to the initial shock.

On the Figure 2.B phase diagram, this intervention would be depicted with a diagonal (with a slope of -1) jump to directly above the steady state, which places it into the relaxed region. From there, the desperate seller population remains at its steady state, while the excess relaxed seller population (including those assets held by the Fed) will gradually draw down. Indeed, the steady state is reached in finite time (at $t = 4.6$).[19]

Figure 4 indicates the consequences of this intervention. For comparison, the solid and dashed lines throughout replicate the results of Figure 3, where the market absorbs the extra desperate sellers on its own. The dotted and dot-dashed lines indicate how these respective paths change due to the intervention.

[19] This finite transition occurs because during the relaxed phase, the relaxed population approaches $\frac{\eta}{\lambda}$, the relaxed steady state, which is strictly below the dispersed steady state.

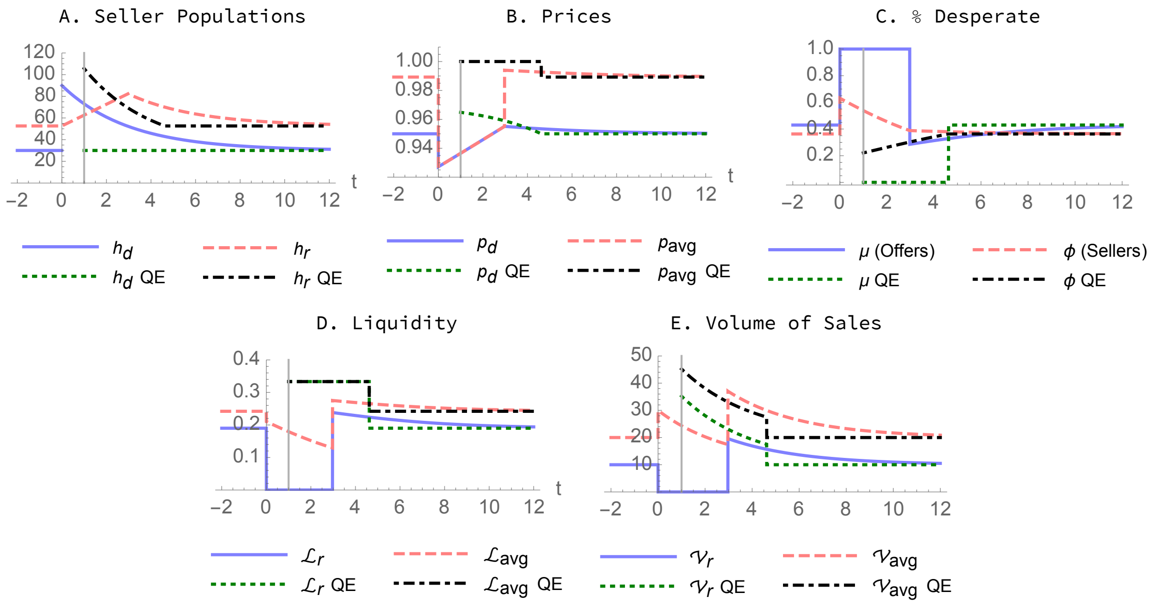

Figure 4: Dynamic response from a 200% increase in the number of desperate sellers at time $t = 0$, followed by an unanticipated QE purchase at time $t = 1$.

In Panel A, we see that the Fed buys 37 assets from desperate sellers, whose population drops to its steady state of 30; and those 37 assets are now counted as if held by relaxed sellers. This abundance of relaxed sellers (dot-dashed line in Panel A) cause buyers to completely reverse their offer strategy (dotted line in Panel C) to exclusively target relaxed sellers. Thus, the stock of relaxed sellers begins to fall instead of rise, and since $\mu = 0$, it falls even faster than it would have in the dispersed phase. Note that some of these sales will be of Fed assets, since the full-offer prices are now available. At $t = 4.6$ the relaxed population reaches its steady state. Without an intervention, a dozen excess sellers of each type (or 40% of the initial infusion of extra sellers) would have yet to transact at $t = 4.6$.

One justification for a QE intervention is to provide greater liquidity for assets. This certainly plays out in the model, facilitating faster transactions in Panel D and higher volumes of transactions in Panel E. Another justification might be to prop up asset prices, which plays out in Panel B. All transactions occur at full price after the intervention but prior to reaching steady state, which is higher than either the desperate- or dispersed-phase prices in the transition without intervention.

At the time of the intervention, the desperate reservation price jumps as they anticipate full price offers for the next 3.6 periods. Thus, the intervention forces the Fed to pay a higher p_d. Thereafter, no one is offered p_d until steady state is reached, but the desperate reservation price gradually declines in anticipation of the return of some desperate price offers.

As a final metric, we compute the average duration of the Fed asset holdings, which is 3.65 periods, only slightly longer than the transition to steady state. We also compute the expected profit from the purchase, discounting the proceeds from the future resale of the asset. While purchased at 3.5% lower than the eventual sale price, the time delay (at a 1% discount rate) exactly offsets this price difference, yielding 0 profits. This result assumes that, like the desperate sellers under our parameters, the Fed receives $x - c = 0$ flow of income while holding the asset. If they derived value from holding the asset, that would become the Fed's profit.

4.2 Experiment 2: Higher Search Frictions

Now consider a permanent decrease in the rate at which sellers encounter buyers.[20] Effectively, this intensifies the search friction, perhaps caused if longer due diligence periods are required for each sale. Figure 5 presents the dynamic response from a 3% decline in λ, where the size of shock was chosen so as to generate the same three-period desperate phase as in the prior experiment.

The new steady state has more sellers of both types, but relatively more relaxed sellers. Indeed, h_d rises imperceptibly, by only 3%. If the new steady state is depicted at the center of the phase portrait in Figure 2.B, the old steady state lies in the lower left quadrant. This leads to an initial desperate phase, during which the population of relaxed sellers quickly climbs as none accepts offers (Figure 5.A), causing ϕ to drop (Figure 5.C). After reaching

[20] A permanent increase in the flow of desperate sellers, δ, produces a nearly identical response, so this can be viewed as an alternative to the temporary increase in the stock of desperate sellers in Experiment 1. If the match rate λ is endogenous, as in Section 5.1, this experiment can be interpreted as a permanent decrease in the search efficiency parameter ψ. The resulting transition is quite similar with Cobb-Douglas matching, and identical if the short side of the market always matches.

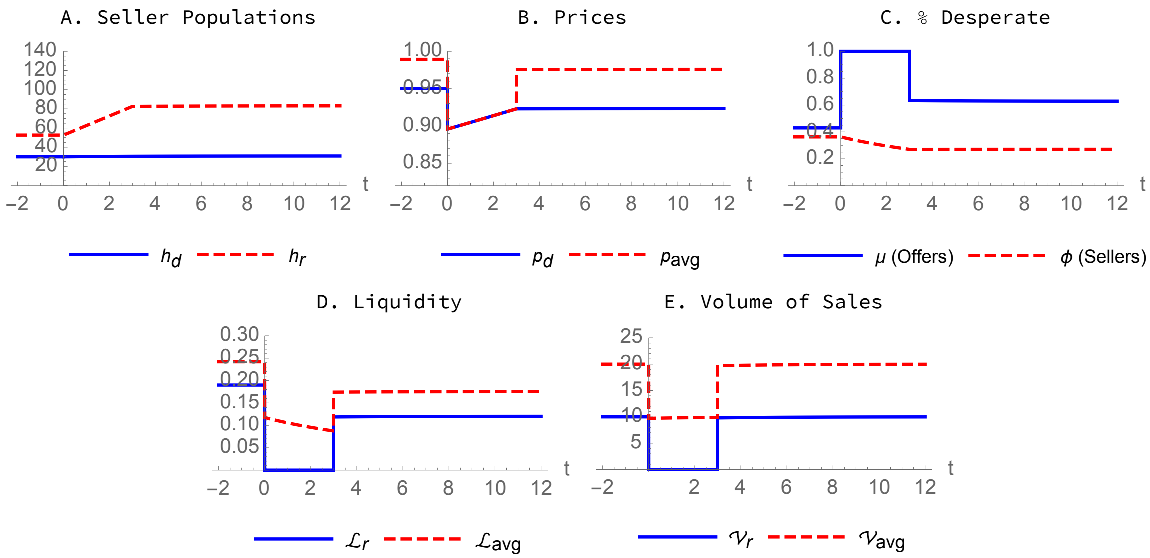

Figure 5: Dynamic response from a permanent 3% decrease in the matching rate λ, starting at time $t = 0$

the critical time $T_d = 3$ where a dispersed phase begins, both populations continue to slowly grow, but the desperate grow slightly faster causing ϕ to rebound slightly. The desperate price drops 5.7% at the time of the shock then slowly recovers by 3% (Figure 5.B).

As shown in the dashed line of Figure 5.D, average liquidity drops precipitously at the time of the shock to half its initial level, then dwindles another 10% over the course of the desperate phase. At $T_d = 3$, average liquidity rebounds 33%, slightly undershooting the new steady state liquidity. As in the first experiment, relaxed liquidity in the solid line completely disappears during the dispersed phase, before slightly undershooting to 35% below the original steady state.

Finally, Figure 5.E demonstrates that aggregate sales volume drops at the time of the shock (as opposed to our first experiment) due to a complete loss of sales from relaxed sellers and no compensating increase in sales from desperate sellers, whose population is unchanged. When price dispersion resumes, sales volume jumps up and approaches its initial level. Although there are more relaxed sellers in the market than in the initial steady state, they receive full-price offers less frequently, which nets the same volume of sales.

In each plot of Figure 5, the dispersed phase looks almost flat, when in fact it continues a monotonic approach to the final steady state. This flat approach merely reflects the fact that at the end of the desperate phase, the populations are already close to the steady state.

The two experiments offer some similarities, yet produce distinct patterns. Both shocks cause a partial freezing of markets and produce a collapse in prices during the initial dispersed phase.[21] In the second experiment, the desperate price gradually and monotonically returns to its new steady state. In the first experiment, however, the desperate price rises beyond its steady state level during the desperate phase, then falls during the dispersed phase. This overshooting occurs whenever the initial condition starts with more desperate sellers than in the steady state. Similarly, in both experiments, average market liquidity discontinuously drops and then continues to slide, but liquidity only overshoots in the first experiment's dispersed phase.

In both experiments, we see that steady state price dispersion is easily disrupted in the short run; indeed, almost any shock will move the economy out of the dispersed phase. This is not surprising since dispersion relies on a mixed strategy, and any small nudge can break that indifference. However, price dispersion is robust in the medium run, being restored in finite time even when the long-run steady state is only asymptotically approached. These theoretical predictions resemble the partial freezing of asset markets after a negative shock; in a desperate phase, sales only occur from desperate sellers who are willing to accept the ubiquitous low prices. Then, at the critical time T_d, the accumulation of relaxed sellers is sufficiently attractive for buyers to make some higher offers again.

[21] Both shocks set initial conditions within the desperate region of Figure 2.A. In our technical appendix available at `https://byu.box.com/v/FireSalesTechnicalAppendix`, we present shocks starting in the relaxed or bifurcated regions. Dynamics are not as rich during a relaxed phase, as the only observed price p_r holds constant. However, this eventually feeds into a dispersed phase (from the relaxed region) or a desperate phase (from the bifurcated region), which are represented in the experiments here.

5 Extensions

5.1 Entry Model

In our baseline model, buyers optimally choose which sellers to target, but only in instantaneous decisions. In this extension, we allow for inter-temporal planning by buyers, disciplining it with competition among buyers. That is, buyers will correctly anticipate the equilibrium path of seller populations and prices, but free entry and exit by the buyers will ensure that expected profits are zero. Relative to our baseline model, this extension endogenizes λ; yet the equilibrium behavior is largely unchanged.

Assume that the rate of meetings in the market follow a Cobb-Douglas matching function[22] with equal weights on buyers and sellers: $m \equiv \psi g^{\frac{1}{2}} h^{\frac{1}{2}}$. Thus, a seller encounters buyers at rate $\lambda \equiv \frac{m}{h}$, while buyers encounter sellers at rate $\alpha \equiv \frac{m}{g}$.

Next, assume that buyers in the market pay a search cost k per period. Let B_r denote the present value of expected profit for a buyer that always offers p_r:

$$\rho B_r = -k + \dot{B}_r + \alpha \left(\frac{y}{\rho} - p_r - B_r \right). \tag{15}$$

Here, time is a state variable because the rate of meeting could change over time. In the last term, meetings occur at rate α, and since the proposed price is always accepted, the buyer will stop searching ($-B_r$) and realize profit of selling ($\frac{y}{\rho} - p_r$).

If the buyer always offers p_d, the present value of expected profit B_d is:

$$\rho B_d = -k + \dot{B}_d + \alpha\phi \left(\frac{y}{\rho} - p_d - B_d \right). \tag{16}$$

As before, ϕ indicates the probability that an offer is rejected (by relaxed sellers).

Since buyers can freely enter either market, we require that $B_r \leq 0$ and $B_d \leq 0$. On the other hand, free exit ensures that no buyer must endure losses. This requirement can

[22] An even simpler extension uses the matching function of $m = \psi \min\{g, h\}$, so that the short side of the market matches at rate ψ while the long side is proportionally rationed. So long as the buyer's search cost is sufficiently small, this environment literally preserves our baseline solution with $\lambda = \psi$.

be expressed as $\mu B_d = 0$ and $(1-\mu)B_r = 0$. We note that relative to our baseline model, instantaneous targeting decision are unaltered (*i.e.* $\Pi > 0$ iff $B_d > B_r$).

The combination of free entry and the matching function allow us to solve for the endogenous λ as:

$$\lambda = \begin{cases} \frac{\psi^2(y-x)}{\rho k} & \text{if } \mu < 1 \\ \frac{\phi\psi^2(y-\rho p_d)}{\rho k} & \text{if } \mu = 1. \end{cases} \tag{17}$$

In a relaxed or dispersed phase ($\mu < 1$), the solution to λ is a constant. The realized profit when a buyer trades with a relaxed seller is constant over time; thus, even when the populations of sellers change, buyers proportionally enter or exit to compete away excess profits. Thus, relaxed and dispersed phases proceed exactly as in the baseline model (under the right ψ and k). However, when all buyers target the desperate sellers ($\mu = 1$), the realized profit falls as price p_d increases over the desperate phase, causing λ to fall over time.

This is also why a full analytic characterization is no longer possible when buyer entry is endogenous. While the relaxed and dispersed phase proceed as before, the desperate population law of motion cannot be analytically solved in the desperate phase. While h_r and p_d can be solved directly and substituted in, the following differential equation in h_d must be numerically solved:

$$\dot{h}_d = \delta - \frac{\psi^2(y-\rho p_d)}{\rho k} \cdot \frac{h_d^2}{h_d + h_r}. \tag{18}$$

The numerical solution bears strong resemblance to the baseline model.[23] In Figure 6, we present a phase portrait for the buyer entry model, using the same parameter values as in Figure 2 and setting $\psi = 1$ and $k = 8.55$ so that the equilibrium $\lambda = 0.33$ in the relaxed or dispersed phases as before.

The relaxed and dispersed phases are literally unchanged from Figure 2.B. A desperate path still occurs in the same region, but has greater curvature than when λ is exogenous.[24]

[23]Indeed, near the dispersed path, we can prove that the behavior is identical to the baseline model.

[24]The same would apply when parameters generate a relaxed steady state, with no change to the relaxed

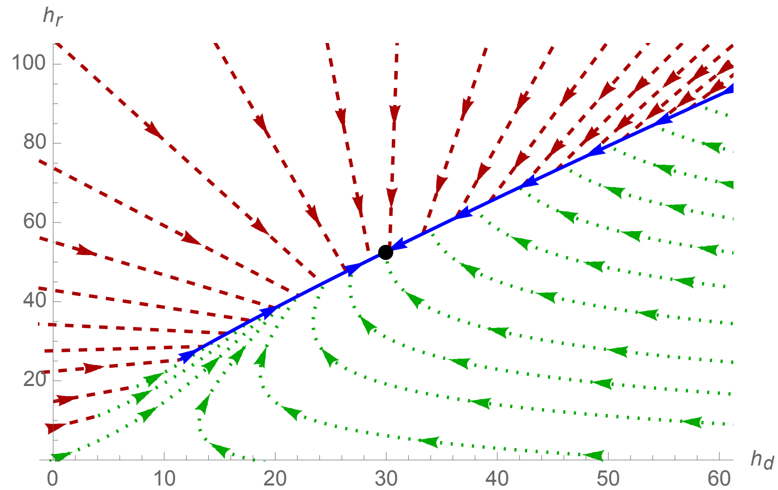

Figure 6: **Phase Portrait with endogenous buyer entry**. The solid dot indicates the dispersed steady state, while the solid line denotes the dispersed phase of the transition path. Dashed and dotted lines indicate the relaxed and desperate phases, respectively. ($\psi = 1$ and $k = 8.55$, with other parameters as in Figure 2)

Indeed, starting with many desperate and few relaxed sellers, enough extra buyers will enter the market to quickly draw down the desperate population, which drops even below the steady state. As this happens, the desperate price climbs until buyers begin leaving the market. This slows down the arrival rate λ and allows h_d to grow toward steady state. The population of relaxed sellers steadily grows as before, since they accept no offers.

5.2 Urgency Model

Our baseline model captures heterogeneity in the marketplace by exogenously assuming that some sellers enter the market with greater urgency to sell. An alternative approach would allow relaxed sellers to become desperate at some point in their search. For instance, a major life event like a wedding or job change may accelerate the need to liquidate assets. We assume this transition randomly arrives at Poisson rate τ. We refer to this as the Urgency model.

This has three effects on our model setup. First, relaxed seller population dynamics

phase or dispersed boundary, but with greater curvature in the desperate region.

now include the exit of those who become desperate at rate τh_r:

$$\dot{h}_r = \eta - \lambda(1-\mu)h_r - \tau h_r. \tag{19}$$

Second, the desperate population is augmented by this flow of relaxed sellers.

$$\dot{h}_d = \delta + \tau h_r - \lambda h_d. \tag{20}$$

Finally, relaxed sellers anticipate this random change in state which occurs at rate τ and moves them from V_r to V_d:

$$\rho V_r = x + \dot{V}_r + \lambda(1-\mu)\left(p_r - V_r\right) + \tau(V_d - V_r). \tag{21}$$

The urgency model offers three potential steady states. Under the right parameters, it is possible to sustain a desperate steady state, in which buyers only offer the desperate price. Relaxed sellers never exit the market due to selling, but eventually become desperate and then make a sale. The three solutions are reported in Table 3, and they are mutually exclusive, resulting in a unique steady state.

For notational simplicity, we introduce the parameter γ, which plays a key role in determining which equilibrium occurs:

$$\gamma \equiv \frac{c\delta\rho - \eta\lambda(y-x)}{\delta\rho^2 + \eta\tau(\lambda - \rho)}. \tag{22}$$

In a dispersed steady state, γ is the difference between prices, $p_r - p_d$.

The key distinction in solving transitions in this model is that the differential equations are interrelated. In particular, relaxed sellers anticipate the possibility of becoming desperate, and thus their reservation price is now affected by market conditions. The resulting

Table 3: Urgency Model: Steady State Solution

	Desperate	Dispersed	Relaxed
	occurs when		
	$\gamma \geq \frac{c}{\rho+\tau}$	$\frac{c}{\rho+\tau} > \gamma > \frac{c}{\rho+\tau+\lambda}$	$\frac{c}{\rho+\tau+\lambda} \geq \gamma$
p_r	$\frac{x}{\rho} - \frac{\tau c}{\rho(\rho+\tau)}$	$\frac{x-\tau\gamma}{\rho}$	$\frac{x}{\rho} - \frac{\tau c}{\rho(\lambda+\rho+\tau)}$
p_d	$\frac{x-c}{\rho}$	$\frac{x-(\rho+\tau)\gamma}{\rho}$	$\frac{x}{\rho} - \frac{c(\rho+\tau)}{\rho(\lambda+\rho+\tau)}$
h_r	$\frac{\eta}{\tau}$	$\frac{\eta\gamma}{c-\rho\gamma}$	$\frac{\eta}{\lambda+\tau}$
h_d	$\frac{\delta+\eta}{\lambda}$	$\frac{\delta c+(\tau\eta-\rho)\gamma}{\lambda(c-\rho\gamma)}$	$\frac{\delta+\eta}{\lambda} - \frac{\eta}{\lambda+\tau}$
μ	1	$1 + \frac{\rho+\tau}{\lambda} - \frac{c}{\lambda\gamma}$	0

system of differential equations simplify to the following:

$$\dot{p}_r = (\tau+\rho)p_r - x - \tau p_d \tag{23}$$

$$\dot{p}_d = \lambda(1-\mu)(p_d - p_r) + \rho p_d - x + c \tag{24}$$

$$\dot{h}_r = \eta - \tau h_r - \lambda(1-\mu)h_r \tag{25}$$

$$\dot{h}_d = \delta + \tau h_r - \lambda h_d \tag{26}$$

These can be directly solved in relaxed or desperate phases ($\mu = 0$ or 1). For a dispersed phase, we can eliminate p_d through substitution, leaving a system of three first-order differential equations (on h_r, h_d, and p_r) which must be numerically solved.

As before, this results in a unique dispersed path.[25] Indeed, the phase portrait of this system bears strong resemblance to Figure 2.B, only adding curvature to each path as τ becomes larger.

[25] Uniqueness comes from computing the eigenvalues of the system. One is always negative while the other two are positive, indicating a saddle point. Thus, if the initial condition of this market is exactly on the saddle path (our dispersed path), it will converge to the steady state. Any other initial condition will diverge eventually, which merely indicates that one cannot sustain a dispersed path throughout the transition; rather, one must start with a relaxed or desperate path until reaching the unique dispersed path.

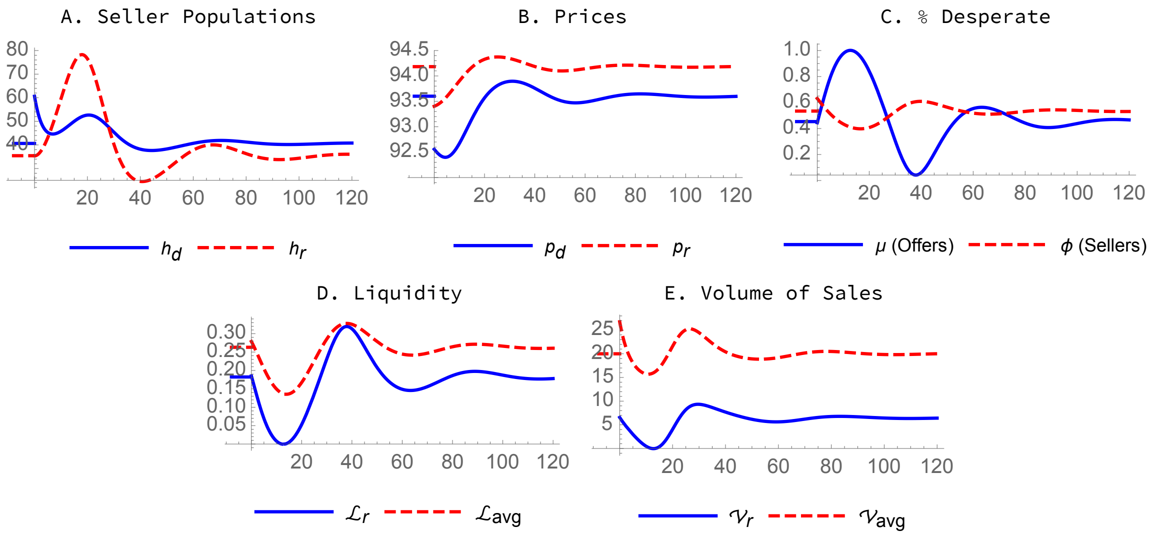

Figure 7: Dynamic response to a 50% increase in desperate sellers at time $t = 0$. ($y = 0.9485$, $c = 0.17$ and $\tau = 0.1$, with other parameters as in Figure 3)

5.3 Reluctant Sellers

Within the urgency model, interesting dynamics can occur when τ is somewhat large and we let $x > y > x - c$. Literally, this means that relaxed sellers currently get a higher instantaneous flow from the asset than the buyers do. Even so, the relaxed sellers are concerned that they may soon become desperate, and are thus willing to sell the asset at a price that is below $\frac{y}{\rho}$. Thus, we refer to them as *reluctant* sellers.

In such a scenario, the dispersed path need not be unique, and within each dispersed path, the transition is no longer monotonic. Indeed, liquidity, prices and populations can oscillate around their eventual steady-state values. We illustrate this in the same context as our first experiment. Suppose that the number of desperate sellers instantaneously increases by 50%. This shock is kept sufficiently small that the market can remain in a dispersed phase throughout the transition. Relative to our first experiment parameterization, the seller's relaxed valuation y is 8 percentage points lower so as to be below the buyer's valuation x. We also introduce a transition rate to desperation of $\tau = 0.1$ per month, and set $c = 0.17$. All other parameters are unchanged. The typical dynamic response illustrated in Figure 7.

Remarkably, each variable's path shows gradually dampening oscillation around the steady state, rather than a monotonic adjustment. The cause of oscillation is fundamentally the same as those driving the non-monotonic transitions seen in our baseline model. The buyers respond to shocks by changing who they target; but by pursuing one type of seller more heavily, buyers unintentionally cause the other type to accumulate.

Before elaborating on this mechanism, it is helpful to note that the oscillations are nearly synchronized across certain variables. The cycles for h_r and μ are coincident, while h_d follows with a slight lag (after its first trough). The cycle for ϕ is nearly coincident but negatively correlated with h_r.

These cycles are largely a matter of who buyers target and its effect on population dynamics. Following the shock, it is not surprising that buyers increasingly target desperate sellers. This causes the desperate price to fall even beyond its initial discrete drop, and decreases liquidity — effectively, the initial shock is propagated to get worse before it gets better. At the same time, relaxed sellers reject more offers, more than doubling their population by the first peak. Buyers are eventually attracted back to targeting relaxed sellers; yet as they do, the pool of relaxed sellers shrinks faster than that of desperate sellers, eventually making the latter more plentiful. This sets up the next cycle of oscillations, as the buyers return their focus to the desperate sellers.[26]

These population and targeting dynamics explain why prices repeatedly overshoot their steady-state levels. Prices reflect the utility of buyers anticipating future market conditions. Thus, p_r remains below steady state whenever μ is rising — fewer full price offers imply a higher chance of becoming desperate before selling the asset. As μ falls, however, relaxed sellers are more likely to obtain a full price offer, giving them a reservation price above steady state. This creates a mismatch between p_r and μ which perpetuates the cycles. Even so, the overall trend pushes the population of sellers back toward the steady-state desperation ratio, thus dampening successive oscillations. That is, the market tends to target and thus draw down whichever seller is relatively abundant, which ultimately dominates any secondary

[26]The mathematical difference between this and the prior extension is that all three eigenvalues are negative around the steady state when $x > y$, leading to a stable spiral toward steady state. We can also combine reluctant sellers with buyer entry from our first extension with no impact on behavior. The zero profit condition again results in a constant λ throughout the dispersed transition.

forces that mismatched timing create.

Dampening oscillations are also seen in the extended model of Kiyotaki and Moore (1997), which can be described with a similar mismatch. Their oscillations depend on asset improvements that are user-specific (thus can't be sold or used as collateral) and lumpy (only a fraction of firms can make improvements per period) and depreciate over time. Oscillation can occur because of mismatch between when financing is available (due to relaxed budget constraints) and when improvements can be made (due to lumpiness). This creates a cycle as firms accumulate, waiting for investment opportunities, then subside when those opportunities come. The cycle dampens as borrowing constraints fade back toward steady state.

Our oscillations dampen, unlike Albrecht, *et al* (2013) and Maurin (2020) which produce an endlessly repeated cycle. These cycles are sustained by the buildup of untraded low-valuation customers or high-quality assets, respectively, followed by a single period in which the market fully clears. Our setting features a similar buildup during excess targeting of low-valuation sellers, but our search friction prevents a full reset to clears the build up; rather it must slowly work out over dampening cycles.

Figure 7 represents one of a continuum of equilibria, each corresponding to an initial value of $p_r(0)$. Multiple equilibria occurs here because of the interdependence of relaxed and desperate utility.[27] The reluctant sellers worry about the desperate price because they may become desperate. The desperate care about the relaxed price because they might be lucky enough to get that offer. Both types must anticipate the path of prices, so each $p_r(0)$ indicates a different anticipated path that will be followed. Even across the multiple equilibria, the qualitative behavior is quite similar after the first cycle.

[27] In the baseline model and prior extensions, the differential equations produce a saddle point at steady state. As a consequence, only one path leads directly to the steady state, while any other diverges. With reluctant sellers, however, the spiral point at steady state can be reached from an open interval of initial conditions $p_r(0)$, each creating a distinct equilibrium.

6 Conclusion

Our model depicts dynamic price formation in an economy where some sellers are more impatient than others to sell their assets. Buyers strategically choose which price they will offer, anticipating that a low price offer will be turned down by a relaxed seller who feels less pressure to liquidate his asset. This model enables us to consider the dynamic transition of seller populations, buyer strategies, asset inventories, offered prices, and liquidity following a shock to the market. In this model, illiquidity stems from a search friction: it takes time to find a buyer who offers an acceptable price given the uncertainty about seller motivations.

We have three main conclusions. First, buyers almost always begin the transition by exclusively targeting one type of seller, even when the eventual steady state involves price dispersion. This seems descriptive of a fire sale, when all buyers insist on bargain prices, while any sellers capable of doing so wait for a market recovery. Second, the dynamic path of prices and liquidity often overshoot their steady state. Exclusively targeting one seller type unintentionally builds up the population of the other type; the resulting imbalance is only relieved once both types are again targeted. Finally, when relaxed sellers can randomly become desperate, prices can oscillate around steady state, slowly dampening over time.

Our theory is applicable to many illiquid asset markets, and has potential application in monetary theory as well. In this simple set-up, we abstract away from the underlying causes of seller distress, because we are interested in a full characterization of the transition path of the economy, which we are able to solve for analytically. We show that strategic buyer targeting is instrumental in explaining an increasing inventory and declining prices during a downturn, as well as the critical point where the recovery starts.

Search frictions provide a natural reason why markets cannot immediately absorb shocks, yet they also impose discipline on the dynamic transition. Transactions are the market's only mechanism to adjust the levels or proportions of sellers, tweaking the rate at which each type exits the market. In our model, buyers effectively control this mechanism in deciding who to target, and use it to maximize their own profits. While the buyers have no direct incentive to restore steady state, they ultimately contribute to the recovery, albeit by a circuitous route.

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A Proofs

Throughout the proofs, let $F(\cdot) \equiv \int_0^1 \frac{(1-s)^{\frac{\rho-2\lambda}{2\lambda}}}{\left(1-s+\frac{\delta}{\lambda \hat{h}_d}s\right)^{\frac{1}{2}}} ds$, so that $\Delta(\hat{h}_d) = \beta \hat{h}_d + \frac{\beta\delta-\eta}{2\lambda}F(\cdot)$. We note that $F(\cdot) > 0$ because the integrand is always positive.

Proof of Proposition 1. The solution for p_r and h_d are immediate from Eqs. 5 and 6, and Eq. 7, respectively. The relaxed population in Eq. 8 reduces to $h_r = \frac{\eta}{\lambda(1-\mu)}$, while the desperate price in Eq. 3 and 4 simplifies to $p_d = \frac{x}{\rho} - \frac{c}{\rho+\lambda(1-\mu)}$.

The difference in expected profit from offering p_d versus p_r is:

$$\frac{h_d}{h_d+h_r}\left(\frac{y}{\rho} - p_d\right) - \frac{y}{\rho} + p_r = \left(1 - \frac{\eta}{\eta+\delta(1-\mu)}\right)\left(\frac{y-x}{\rho} + \frac{c}{\lambda(1-\mu)+\rho}\right) - \frac{y-x}{\rho}.$$

After substituting for $y-x$ using the expression for β, this becomes:

$$\frac{c(\beta\delta(1-\mu)(\lambda+\rho) - \eta(\lambda(1-\mu)+\rho))}{\beta(\lambda+\rho)(\delta(1-\mu)+\eta)(\lambda(1-\mu)+\rho)}.$$

Note that the denominator is always positive. When $\mu = 1$, the numerator becomes $-c\eta\rho < 0$, meaning that offering the relaxed price is strictly more profitable, contradicting everyone offering the desperate price. This rules out a desperate steady state. When $\mu = 0$, the numerator becomes $c(\lambda+\rho)(\beta\delta-\eta)$, which is weakly negative (and thus consistent with a relaxed steady state) iff $\beta\delta \leq \eta$. Lastly, note that the numerator is linear in μ, implying a single $\mu = 1 - \frac{\rho\eta}{(\rho+\lambda)(\beta\delta-\eta)+\rho\eta}$ where the numerator equals 0. This $\mu > 0$ iff $\beta\delta > \eta$. □

Proof of Corrolary 1. Note that $\partial\beta/\partial\lambda = -\frac{\rho c}{(y-x)(\rho+\lambda)^2} < 0$. Indeed, $\lim_{\lambda\to\infty}\beta = 0$. Thus, as λ increases, β will eventually fall below η/δ, switching from a dispersed to a relaxed steady state. The stated condition is merely a rearrangement of $\beta\delta > \eta$.

In the dispersed steady state, $\partial p_d/\partial\lambda = \frac{\eta(y-x)}{\delta\rho^2} > 0$, $\partial h_d/\partial\lambda = -\frac{\delta}{\lambda^2} < 0$, and $\partial h_r/\partial\lambda = -\frac{\delta c}{(y-x)\lambda^2} < 0$.

In the relaxed steady state, $\partial p_d/\partial\lambda = \frac{\eta(y-x)}{\delta\rho^2} > 0$, $\partial h_d/\partial\lambda = -\frac{\delta}{\lambda^2} < 0$, and $\partial h_r/\partial\lambda = -\frac{\delta c}{(y-x)\lambda^2} < 0$. □

Proof of Lemma 1. First, consider p_r. If we substitute Eq. 6 into Eq. 5, we find that at any time, the following must hold:

$$\rho p_r = x + \dot{p}_r. \tag{27}$$

This differential equation only has one solution that converges: $p_r = \frac{x}{\rho}$ for all time.

Next, consider the h_d. The differential Eq. 7 with initial condition $h_d(0) = \frac{\delta}{\lambda} + q_d$ has $h_d(t) = \frac{\delta}{\lambda} + q_d e^{-\lambda t}$ as its unique solution.

For the h_r in either degenerate phase, the differential Eq. 8 with initial condition $h_r(0) = \frac{\eta}{\lambda} + q_r$ results in a unique solution $h_r(t) = \frac{\eta}{\lambda} + q_r + \eta t$ when $\mu = 1$ is inserted or a unique solution $h_r(t) = \frac{\eta}{\lambda} + q_r e^{-\lambda t}$ when $\mu = 0$ is inserted.

To obtain the p_d in a desperate phase, substitute $\mu = 1$ and Eq. 4 into Eq. 3 to obtain $\rho p_d = x - c + \dot{p}_d$. This differential equation has $\frac{x-c}{\rho} + a_d e^{\rho t}$ as its unique solution (up to the constant a_d). The same process in a relaxed phase ($\mu = 0$) obtains $\rho p_d = x - c + \dot{p}_d + \lambda \left(\frac{x}{\rho} - p_d\right)$, which has $\frac{x-\beta(y-x)}{\rho} + a_r e^{t(\rho+\lambda)}$ as its unique solution (up to the constant a_r).

For the dispersed phase, we first obtain $p_d(t) = \frac{x}{\rho} - \frac{y-x}{\rho} \cdot \frac{h_r(t)}{h_d(t)}$ from rearrangement of the equal profit condition, ensuring that $\Pi(t) = 0$ throughout this phase. We then substitute this and its derivative into the pricing differential equation, $\rho p_d = x - c + \dot{p}_d + \lambda(1 - \mu)\left(\frac{x}{\rho} - p_d\right)$, and solve for μ. This allows us to eliminate μ in Eq. 8, the law of motion h_r, which has the unique solution $h_r = \Delta(h_d) + a_p\sqrt{\lambda h_d} e^{\frac{\rho t}{2}}$ up to a constant a_p.

If the system remains in a dispersed phase indefinitely, then a_p must equal zero; otherwise $h_r \to \pm\infty$. Indeed, the system cannot change to a different dispersed phase (*e.g.* a distinct constant a_p) at some time T because the solution to h_d is independently determined and hence h_r would discretely jump at T. Moreover, the system cannot shift to a desperate or relaxed phase at some finite time T, because by construction the dispersed path maintains equal profits between both price offers. If instead one group is exclusively targeted, the other group becomes disproportionately plentiful and thus strictly more profitable. Thus, the dispersed phase continues indefinitely after it begins with $a_p = 0$, yielding the solution $h_r(t) = \Delta(h_d(t))$. □

Proof of Proposition 2. We begin by establishing that the proposed path is an equilibrium. For Case 1, suppose $h_r(0) \geq \beta h_d(0)$. From Lemma 1, we need only to establish that $\Pi(t) \leq 0$.

First, we verify that $h_r(t) \geq \beta h_d(t)$ for all t. After substituting for $h_r(t)$ and $h_d(t)$ with the proposed solutions and rearranging, this is equivalent to

$$\eta - \beta\delta \geq \lambda\left(\beta q_d - q_r\right) e^{-\lambda t}.$$

If $\beta q_d \leq q_r$, then $\lambda\left(\beta q_d - q_r\right) e^{-\lambda t} \leq 0$ for all t while $\eta - \beta\delta \geq 0$ by assumption, so the inequality holds. If $\beta q_d > q_r$, recall that the inequality must hold at $t = 0$ (because it is equivalent to $h_r(0) \geq \beta h_d(0)$), and $\lambda\left(\beta q_d - q_r\right) e^{-\lambda t}$ is decreasing in t. Therefore the inequality holds for all t. Thus, $h_r(t) \geq \beta h_d(t)$ for all t.

Thus, profits from offering the desperate price are always negative:

$$\Pi(t) = \frac{y-x}{\rho}\left((1+\beta)\frac{h_d(t)}{h_r(t) + h_d(t)} - 1\right) = \frac{y-x}{\rho(h_r(t) + h_d(t))}(\beta h_d(t) - h_r(t)) \leq 0.$$

Moving to Case 2, suppose that $h_r(0) < \beta h_d(0)$. We first establish that there is a unique time T on the proposed path for h_d and h_r such that $h_r(T) = \beta h_d(T)$, or by substitution:

$$\frac{\eta}{\lambda} + q_r + \eta T = \beta \left(\frac{\delta}{\lambda} + q_d e^{-\lambda T} \right).$$

The l.h.s. has slope η w.r.t. T, while the r.h.s. has slope $-\beta \lambda q_d e^{-\lambda T}$. If $q_d \geq 0$, then the r.h.s. is always decreasing, creating a unique T where the two sides equate. If $q_d < 0$, then the r.h.s. is increasing and at an increasing rate ($\beta \lambda^2 q_d e^{-\lambda T}$); thus, it will eventually equate with the l.h.s. but cannot intersect again thereafter. Thus, a unique T exists such that $h_r(T) = \beta h_d(T)$.

For $t \geq T$, the Case 1 analysis applies because $h_r(t) \geq \beta h_d(t)$. We need to establish that $\Pi(t) \geq 0$ for $t < T$ in order to apply Lemma 1.

First, suppose that $q_d \geq 0$. By Lemma 1 it follows that $h_d(t) \geq \delta/\lambda$ and $h_d'(t) \leq 0$ for all $t < T$. Note that the following has the same zeros and sign as the profit function:

$$\tilde{\Pi}(t) \equiv \Pi(t) \cdot \left(1 + \frac{h_r(t)}{h_d(t)} \right) = \frac{x}{\rho} - p_d(t) - \frac{y-x}{\rho} \cdot \frac{h_r(t)}{h_d(t)}.$$

The derivative of this profit function is:

$$\tilde{\Pi}'(t) = -p_d'(t) - \frac{y-x}{\rho} \cdot \frac{h_d(t) h_r'(t) - h_r(t) h_d'(t)}{h_d(t)^2} < 0.$$

The inequality holds because $p_d'(t) = \frac{\lambda c}{\rho(\rho+\lambda)} e^{\rho(t-T)} > 0$ and $h_r'(t) = \eta > 0$, while $h_d'(t) \leq 0$. Since $\tilde{\Pi}(T) = 0$, this ensures that $\tilde{\Pi}(t) < 0$ for all $t < T$.

Next, suppose that $q_d < 0$, implying that $h_d(t) < \delta/\lambda$. Here, we use the following equation which has the same zeros and sign as the profit function:

$$\hat{\Pi}(t) \equiv \Pi(t) \cdot (h_d(t) + h_r(t)) = \left(\frac{y}{\rho} - p_d(t) \right) h_d(t) - \frac{y-x}{\rho} \left(h_d(t) + h_r(t) \right).$$

Using $\beta h_d(T) = h_r(T) = \frac{\eta}{\lambda} + q_r + \eta T$, we substitute for q_r to obtain $h_r(t) = \beta h_d(T) + \eta(t-T)$. Similarly, we use $h_d(T) = \frac{\delta}{\lambda} + q_d e^{-\lambda T}$ to substitute for q_d to get $h_d(t) = \frac{\delta}{\lambda} + \left(h_d(T) - \frac{\delta}{\lambda} \right) e^{\lambda(T-t)}$. We then take the inverse of this latter function, solving for t as a function of h_d and substituting for t in $\hat{\Pi}$, thereby expressing profit entirely in terms of h_d:

$$\hat{\Pi}(h_d) = \frac{c h_d \lambda^2 \left(1 - \left(\frac{\delta - \lambda h_d(T)}{\delta - \lambda h_d} \right)^{\rho/\lambda} \right) + c \lambda \rho (h_d - h_d(T)) - \eta(\lambda+\rho)(y-x) \ln \left(\frac{\delta - \lambda h_d(T)}{\delta - \lambda h_d} \right)}{\lambda \rho (\lambda + \rho)}.$$

Note that $\hat{\Pi}(h_d(T)) = 0$. Moreover, if the relaxed population is at its lowest possible level of $h_r = 0$, the corresponding desperate population on this path would be $\bar{h}_d = \frac{\delta}{\lambda} + \left(h_d(T) - \frac{\delta}{\lambda}\right) e^{\frac{\beta\lambda h_d(T)}{\eta}}$. Evaluated at that point, profit would be:

$$\hat{\Pi}(\bar{h}_d) = \frac{c}{\rho(\lambda+\rho)} \left(\rho + \lambda \left(1 - e^{-\frac{c\rho^2 h_d(T)}{\eta(\lambda+\rho)(y-x)}}\right)\right) > 0.$$

Finally, the second derivative of profit for any $h_d \in [\bar{h}_d, h_d(T)]$ is:

$$\hat{\Pi}''(h_d) = -\lambda\rho(\lambda+\rho)(\delta-\lambda h_d)^2 \left(c\rho(2\delta + (\rho-\lambda)h_d)\left(\frac{\delta - \lambda h_d(T)}{\delta - \lambda h_d}\right)^{\frac{\rho}{\lambda}} + \eta(\lambda+\rho)(y-x)\right) < 0.$$

Under the supposition that $q_d < 0$, we know that $h_d < \delta/\lambda$, so the last parenthetical term is always positive. Thus, if $\hat{\Pi}(h_d) = 0$ for some $h_d < h_d(T)$, it would have $\hat{\Pi}'(h_d) < 0$ at that h_d, and it could not later increase to reach $\hat{\Pi}(h_d(T)) = 0$. Thus $\hat{\Pi}(h_d) > 0$ for all $h_d < h_d(T)$.

We now demonstrate that the equilibrium path is unique. Lemma 1 establishes possible paths, so within that set, we verify three facts: a_r and a_d are unique, the transition must conclude in a relaxed phase, and a dispersed phase cannot occur.

First, note that if $a_r \neq 0$ then $p_d(t) \to \pm\infty$ as $t \to \infty$, which eventually violates $\Pi(t) < 0$. Similarly, in the second case, $a_d \neq \frac{\lambda c}{\rho(\rho+\lambda)} e^{-\rho T}$ is necessary to obtain $h_r(T) = \beta h_d(T)$. With a smaller a_d, buyers prefer to offer desperate prices beyond time T; with a larger a_d, buyers prefer relaxed prices before time T. Either way, buyers are making suboptimal offers and/or sellers are incorrectly anticipating the evolution of prices in the market.

Second, the equilibrium path cannot approach steady state concluding in a desperate phase. To do so, the same constant a_d is required for the desperate price to reach its steady state level (with perhaps a different T). However, any other T would violate the buyer's optimization again, as in the previous paragraph.

Finally, a dispersed phase cannot occur. When $h_r(t) < \beta h_d(t)$ or $h_r(t) > \beta h_d(t)$, buyers strictly prefer offering one price and thus cannot mix. When $h_r(t) = \beta h_d(t)$, offering either price is equally profitable, but only for an instant because h_r would grow faster than h_d. This necessarily pushes $h_r(t+\epsilon) > \beta h_d(t+\epsilon)$ for any $\epsilon > 0$, making p_d offers unprofitable the next instant. □

Lemma 2. *There is a unique solution $H > 0$ to Equation 14. It always satisfies $H < \frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}$. Moreover, in a dispersed phase where $h_r = \Delta(h_d)$, then $\mu(h_d) \in [0,1)$ if $h_d > H$, $\mu(h_d) = 1$ if $h_d = H$, and $\mu(h_d) > 1$ if $h_d < H$.*

Proof of Lemma 2. The derivative of the dispersed path $h_r = \Delta(h_d)$ can be expressed as:

$$\Delta'(h_d) = \frac{\beta(\rho+\lambda)h_d^2 - \eta h_d - (\delta + (\rho-\lambda)h_d)\Delta(h_d)}{2(\lambda h_d - \delta)h_d}. \tag{28}$$

If we substitute for t as Proposition 2, the desperate path can be expressed as $h_r = \beta h_d(T) + \frac{\eta}{\lambda}\ln\frac{\delta-\lambda h_d(T)}{\delta-\lambda h_d}$. If we take its derivative w.r.t. h_d evaluated as $h_d \to h_d(T)$ (approaching the dispersed path), we obtain $\frac{\eta}{\delta-\lambda h_d}$.

The dispersed and desperate paths have the same slope when $h_r = Q(h_d) \equiv \frac{\eta(y-x)+\rho c h_d}{(y-x)(\delta+h_d(\rho-\lambda))}h_d$. Note that Eq. 14 can be written $\Delta(H) = Q(H)$.

The dispersed path has $\mu(t) = 1 - \frac{\eta - h_r'(t)}{\lambda h_r(t)}$. We note that $h_r'(t) = \Delta'(h_d(t))h_d'(t)$, and $h_d'(t) = \delta - \lambda h_d(t)$ from Eq. 7. Thus, $\mu(h_d) = 1 - \frac{\eta - \Delta'(h_d)(\delta-\lambda h_d)}{\lambda\Delta(h_d)}$. Thus, $\mu(h_d) = 1$ iff $\Delta'(h_d) = \frac{\eta}{\delta-\lambda h_d}$, or equivalently, $\Delta(h_d) = Q(h_d)$.

To show that the solution for $H > 0$ is unique, note that $\Delta(0) = 0$ and $Q(0) = 0$. However, $\Delta'(0) = \infty$ while $Q'(0) = \frac{\eta}{\delta}$, so $\Delta(\epsilon) > Q(\epsilon)$ for $\epsilon > 0$ sufficiently small. Moreover, $Q(h_d) \to +\infty$ as $h_d \to_+ \frac{\delta}{\lambda-\rho}$, but $Q(h_d) < 0$ for all $h_d > \frac{\delta}{\lambda-\rho}$. On the other hand, $\Delta(h_d) > 0$ for all $h_d > 0$. Thus, any intersection H must occur at $H < \frac{\delta}{\lambda-\rho}$.

To show that $H < \frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}$, we show that $Q\left(\frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}\right) > \Delta\left(\frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}\right)$. After substitution of $\Delta(\hat{h}_d) = \beta\hat{h}_d + \frac{\beta\delta-\eta}{2\lambda}F(\cdot)$ and rearrangement, this equivalent to:

$$(2\lambda - \rho F(\cdot))(\delta c - (y-x)\eta) + \eta\lambda(y-x)F(\cdot) > 0.$$

Recall that $F(\cdot) > 0$. Moreover, $\frac{(1-s)^{\frac{\rho-2\lambda}{2\lambda}}}{\left(1-s+\frac{\delta}{\lambda h_d}s\right)^{\frac{1}{2}}} \leq (1-s)^{\frac{\rho-2\lambda}{2\lambda}}$ for all $h_d \leq \frac{\delta}{\lambda}$ and all $s \in [0,1]$, so:

$$F(\cdot) \leq \int_0^1 (1-s)^{\frac{\rho-2\lambda}{2\lambda}}ds = \frac{2\lambda}{\rho}.$$

Therefore, the first parenthesis is positive and the inequality holds. Thus, the intersection of Q and Δ must occur at $H < \frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}$.

Meanwhile, $Q''(h_d) = \frac{2\delta(c\delta\rho+\eta(\lambda-\rho)(y-x))}{(y-x)(\delta+(\rho-\lambda)h_d)^3} > 0$ for all $h_d < \frac{\delta}{\lambda-\rho}$. However,

$$\Delta''(h_d) = -\frac{\beta\delta-\eta}{2\lambda}\int_0^1 \frac{\delta t(1-s)^{\frac{\rho-2\lambda}{2\lambda}}(\delta s + 4h_d\lambda(1-s))}{4h_d^4\lambda^2\left(\frac{\delta s}{h_d\lambda} - s + 1\right)^{5/2}}ds < 0,$$

where the inequality holds because $s < 1$ and $\beta\delta > \eta$. Thus, $Q(H) = \Delta(H)$ exactly once for some $H > 0$.

This also means that $Q(h_d) < \Delta(h_d)$ iff $h_d < H$. Moreover, if $Q(h_d) < \Delta(h_d)$ then $\Delta'(h_d) > \frac{\eta}{\delta-\lambda h_d}$ and $\mu(h_d) > 1$. Conversely, $h_d > H$ implies $\mu(h_d) < 1$.

Finally, note that by substituting for $\Delta'(h_d)$ using Eq. 28, we find that $\mu = 1 -$

$\frac{\eta-\Delta'(h_d)(\delta-\lambda h_d)}{\lambda\Delta(h_d)} > 0$ is equivalent to:

$$\lambda h_d \Delta(h_d)\left(\eta h_d - \delta\Delta(h_d) + h_d(\lambda+\rho)(\beta h_d - \Delta(h_d))\right) < 0.$$

This holds for all $h_d > 0$ since $\Delta(h_d) > \beta h_d$ and $\eta < \beta\delta$. Thus $\mu(h_d) > 0$ for all h_d.

□

Lemma 3. *If $\beta\delta > \eta$ and $h_d > H$, then $ch_d > (y-x)\Delta(h_d)$.*

Proof of Lemma 3. First, we show that $cH-(y-x)\Delta(H) > 0$. Substituting $\Delta(H) = Q(H)$, this is equivalent to:

$$\frac{c(\delta-\lambda H)-\eta(y-x)}{\delta-(\lambda-\rho)H}H > 0$$

The denominator is positive, since $H < \frac{\delta}{\lambda}$, while the numerator is positive because $H < \frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}$. In addition, since $\Delta'(H) = \frac{\eta}{\delta-\lambda H}$, the derivative of $cH-(y-x)\Delta(H)$ w.r.t. H is:

$$c-(y-x)\Delta'(H) = \frac{c(\delta-\lambda H)-(y-x)\eta}{\delta-\lambda H} > 0.$$

We also note that at the steady state $h_d = \frac{\delta}{\lambda}$, $ch_d-(y-x)\Delta(h_d) = \frac{(y-x)\eta}{\rho} > 0$ and $c-(y-x)\Delta'(h_d) = \frac{\lambda(\rho\delta c+\eta\lambda(y-x))}{\delta\rho(2\lambda+\rho)} > 0$.

Next, we demonstrate that $c > (y-x)\Delta'(h_d)$ for all $h_d \in \left(H, \frac{\delta}{\lambda}\right)$. Substituting for β and for $\Delta'(h_d)$ using Eq. 28 and rearranging:

$$\frac{c(2\delta-(2\lambda-\rho)h_d)-\eta(y-x)}{(y-x)(\delta-(\lambda-\rho)h_d)}h_d > \Delta(h_d). \tag{29}$$

Since $\Delta'(h_d) > 0$, $\Delta(h_d) < \Delta\left(\frac{\delta}{\lambda}\right)$ for all $h_d < \frac{\delta}{\lambda}$. Thus, Eq. 29 holds if

$$\frac{c(2\delta-(2\lambda-\rho)h_d)-\eta(y-x)}{(y-x)(\delta-(\lambda-\rho)h_d)}h_d > \Delta\left(\frac{\delta}{\lambda}\right).$$

Since $\Delta\left(\frac{\delta}{\lambda}\right) = \frac{\delta c}{(y-x)\lambda} - \frac{\eta}{\rho}$, this rearranges to $h_d > \frac{\rho\delta c-\eta\lambda(y-x)}{c\rho(2\lambda-\rho)}$. Thus, $c > (y-x)\Delta'(h_d)$ for all $h_d \in \left(\frac{\rho\delta c-\eta\lambda(y-x)}{c\rho(2\lambda-\rho)}, \frac{\delta}{\lambda}\right)$.

Moreover, since $\Delta(h_d) < Q(h_d)$ if $h_d \in \left(H, \frac{\delta}{\lambda}\right)$, the inequality also holds if

$$\frac{c(2\delta-(2\lambda-\rho)h_d)-\eta(y-x)}{(y-x)(\delta-(\lambda-\rho)h_d)}h_d > Q(h_d) \iff (\delta-(\lambda-\rho)h_d)(c(\delta-\lambda h_d)-\eta(y-x)) > 0.$$

Thus, $c > (y-x)\Delta'(h_d)$ for all $h_d \in \left(H, \frac{\delta c-\eta\rho(y-x)}{\lambda c}\right)$.

These intervals overlap, since:

$$\frac{\rho\delta c - \eta\lambda(y-x)}{c\rho\,(2\lambda-\rho)} < \frac{\delta c - \eta\rho(y-x)}{\lambda c} \quad \Longleftrightarrow$$

$$c\left(c\delta\rho(\lambda-\rho) + \eta\left(\lambda(\lambda-2\rho^2)+\rho^3\right)(y-x)\right) > 0.$$

The inequality holds because $\lambda > 2\rho$ and $\rho < 1$ by assumption.

Next, consider $h_d \in \left(\frac{\delta}{\lambda}, \frac{\delta}{\lambda-\rho}\right)$. In this case, the rearrangement of the derivative $c-(y-x)\Delta'(h_d)$ reverses the sign of Eq. 29:

$$\frac{c(2\delta-(2\lambda-\rho)h_d)-\eta(y-x)}{(y-x)(\delta-(\lambda-\rho)h_d)}h_d < \Delta(h_d). \tag{30}$$

The derivative of the l.h.s. w.r.t. h_d is:

$$\frac{c\left(h_d^2\rho^2 + h_d\rho(2\delta-3h_d\lambda)+2(\delta-h_d\lambda)^2\right)-\delta\eta(y-x)}{(y-x)(\delta+h_d(\rho-\lambda))^2}. \tag{31}$$

The derivative of this, in turn, is:

$$-\frac{2\delta(c\delta\rho+\eta(y-x)(\lambda-\rho))}{(y-x)(\delta+h_d(\rho-\lambda))^3} < 0.$$

Its denominator is positive since $h_d < \frac{\delta}{\lambda-\rho}$. The numerator is positive because $\lambda > \rho$. Thus, the largest value of Eq. 31 occurs at $h_d = \frac{\delta}{\lambda}$, which yields:

$$-\frac{c\delta\rho(\lambda-\rho)+\eta\lambda^2(y-x)}{\delta\rho^2(y-x)} < 0.$$

Thus, the l.h.s. of Eq. 30 is decreasing for all $h_d \in \left(\frac{\delta}{\lambda}, \frac{\delta}{\lambda-\rho}\right)$. We have previously established that $\Delta'(h_d) > 0$. Thus, the inequality which holds at $h_d = \frac{\delta}{\lambda}$ is relaxed further as h_d increases. Thus, $c-(y-x)\Delta'(h_d)$ for all $h_d \in \left(\frac{\delta}{\lambda}, \frac{\delta}{\lambda-\rho}\right)$.

Finally, when $h_d > \frac{\delta}{\lambda-\rho}$, the rearrangement of $c-(y-x)\Delta'(h_d)$ recovers Eq. 29. In this interval, we proceed by finding a lower bound on the l.h.s. and an upper bound on the r.h.s., then show that these former is greater than the latter. The l.h.s. is bounded below as follows:

$$\frac{c(2\delta-(2\lambda-\rho)h_d)-\eta(y-x)}{(y-x)(\delta-(\lambda-\rho)h_d)}h_d > \frac{(c(2\lambda-\rho))}{(\lambda-\rho)(y-x)}h_d.$$

which holds because this is equivalent to:

$$(\lambda-\rho)(y-x)((\lambda-\rho)h_d-\delta)(c\delta\rho+\eta(\lambda-\rho)(y-x))h_d > 0.$$

Since $\Delta''(h_d) < 0$, the r.h.s. of Eq. 29, $\Delta(h_d)$, is bounded above:

$$\Delta(h_d) < \frac{c\delta\rho(\lambda+\rho) - \eta\lambda^2(y-x)}{\delta\rho(2\lambda+\rho)(y-x)} h_d + \frac{c\delta\rho - \eta(\lambda+\rho)(y-x)}{\rho(2\lambda+\rho)(y-x)}.$$

This inequality holds because both sides are equal and share the same slope at $h_d = \frac{\delta}{\lambda}$, but the r.h.s. has a constant slope, while the slope of l.h.s. shrinks as h_d increases.

We then compare these two bounds:

$$\frac{(c(2\lambda-\rho))}{(\lambda-\rho)(y-x)} h_d > \frac{c\delta\rho(\lambda+\rho) - \eta\lambda^2(y-x)}{\delta\rho(2\lambda+\rho)(y-x)} h_d + \frac{c\delta\rho - \eta(\lambda+\rho)(y-x)}{\rho(2\lambda+\rho)(y-x)},$$

which simplifies to

$$\frac{\delta(\lambda-\rho)(c\delta\rho - \eta(\lambda+\rho)(y-x))}{\lambda^2(3c\delta\rho + \eta(\lambda-\rho)(y-x))} < h_d.$$

At the smallest $h_d = \frac{\delta}{\lambda}$, this becomes $(2\lambda+\rho)(\eta(\lambda-\rho)(y-x) + c\delta\rho) > 0$. Thus, $c - (x - y)\Delta'(h_d) > 0$ for all $h_d > \frac{\delta}{\lambda-\rho}$.

Combined, this establishes that $c - (x-y)\Delta'(h_d) > 0$ for all $h_d \geq H$. Moreover, since $cH - (x-y)\Delta(H) > 0$, this ensures that $ch_d > (x-y)\Delta(h_d)$ for all $h_d > H$. □

Proof of Proposition 3. Consider Case 1. Note that the relaxed path described in Eq. 12 is found by solving $h_d = \frac{\delta}{\lambda} + \left(\hat{h}_d - \frac{\delta}{\lambda}\right) e^{-\lambda t}$ for t and substituting it into $h_r = \frac{\eta}{\lambda} + \left(\Delta(\hat{h}_d) - \frac{\eta}{\lambda}\right) e^{-\lambda t}$. The desperate path in Eq. 13 is found by similarly substituting for t in $h_r = \hat{h}_r + \eta t$.

If $h_r(0) = \Delta(h_d(0))$ and $h_d(0) \geq H$, then Lemmas 1 and 2 apply. Both prices are equally profitable throughout the dynamic transition, so the dispersed equilibrium can be maintained.

This likewise applies in the other three cases after the dispersed phase is reached. For instance, by definition, $T_r \equiv \frac{1}{\lambda} \ln \frac{\lambda q_d}{\lambda \hat{h}_d - \delta}$, which implies that $h_d(T_r) = \hat{h}_d$ per Table 2. Because $\hat{h}_d$ satisfies $h_r(0) = R\left(h_d(0), \hat{h}_d\right)$, this relaxed path ensures that $h_r(T) = \Delta(\hat{h}_d)$. Thus, $h_r(T_r) = \Delta(h_d(T_r))$ and $h_d(T_r) = \hat{h}_d \geq H$. The same reasoning applies to T_d and $\hat{h}_d$ in the third and fourth cases.

Next, as in Case 2, suppose $h_r(0) > \Delta(h_d(0))$ and $\hat{h}_d \geq H$. To show there exists a unique solution for $\hat{h}_d$, first note that $h_r(0) = R(h_d(0), \hat{h}_d)$ rearranges to:

$$\Psi(\hat{h}_d) \equiv \Delta(\hat{h}_d) - \frac{(\delta - \lambda\hat{h}_d)h_r(0) + \eta(\hat{h}_d - h_d(0))}{\delta - \lambda h_d(0)} = 0, \tag{32}$$

with a domain for Ψ of $\left[h_d(0), \frac{\delta}{\lambda}\right]$, since $\lim_{t\to\infty} \frac{\delta}{\lambda} + q_d e^{-\lambda t} = \frac{\delta}{\lambda}$. By assumption, $\Psi(h_d(0)) =$

$\Delta(h_d(0)) - h_r(0) < 0$. On the other hand,

$$\Psi\left(\frac{\delta}{\lambda}\right) = \Delta\left(\frac{\delta}{\lambda}\right) - \frac{\eta}{\lambda} = \frac{\delta c}{(y-x)\lambda} - \frac{\eta(\rho+\lambda)}{\rho\lambda} = \beta\delta - \eta > 0.$$

Since Ψ is a continuous function, there exists an $\hat{h}_d$ such that $\Psi(\hat{h}_d) = 0$. Moreover, the second derivative $\Psi''(\hat{h}_d) = \Delta''(\hat{h}_d)$ is strictly negative, as shown in the proof of Lemma 2. Therefore, if $\Psi'(h_d) \leq 0$ for any $h_d \leq \frac{\delta}{\lambda}$, then $\Psi(h_d) \geq \Psi\left(\frac{\delta}{\lambda}\right)$. Since $\Psi\left(\frac{\delta}{\lambda}\right) > 0$, then $\Psi(h_d) = 0$ can only occur when $\Psi'(h_d) > 0$. Thus, $\Psi(h_d)$ only crosses zero once, yielding a unique solution for $\hat{h}_d$ and the corresponding T_r. This also assures us that $h_r(t) > \Delta(h_d(t))$ for all $t < T_r$. In addition,

$$\Delta(h_d(t)) = \beta h_d(t) + \frac{\beta\delta - \eta}{2\lambda} F(\cdot) > \beta h_d(t),$$

for all t because $\delta\beta > \eta$ and $F(\cdot)$ is always positive.

We now show that profits are negative throughout the relaxed phase, using the rearranged profit function $\tilde{\Pi}(t) = \frac{x}{\rho} - p_d(t) - \frac{y-x}{\rho} \cdot \frac{h_r(t)}{h_d(t)}$ from Proposition 2, which shares the same signs and zeros as $\Pi(t)$. This has a first derivative of:

$$\begin{aligned} \tilde{\Pi}'(t) &= -(y-x) \cdot \left(\frac{h_r(t)}{h_d(t)}\right)' - \rho p_d'(t) \\ &= -(y-x) \cdot \frac{\eta h_d(t) - \delta h_r(t)}{h_d(t)^2} - \rho a_r(\rho+\lambda) e^{(\rho+\lambda)t}. \end{aligned}$$

Since $\Delta(\hat{h}_d) > \beta\hat{h}_d$, we know $a_r < 0$. Moreover, $h_r(t) > \beta h_d(t)$ and $\delta\beta > \eta$ imply $\eta h_d(t) < \delta h_r(t)$. Therefore $\tilde{\Pi}'(t) > 0$ for $t < T_r$. Since $\tilde{\Pi}(T_r) = 0$, then $\tilde{\Pi}(t) < 0$ and $\Pi(t) < 0$ for $t < T_r$.

Next, as in Case 3, suppose $h_r(0) < \Delta(h_d(0))$ and $h_r(0) < \frac{c-a_d}{y-x} h_d(0)$. First, we note that $\hat{h}_d$ is unique. To see this, consider the derivative $S_{\hat{h}_d} = \Delta'(\hat{h}_d) + \frac{\eta}{\lambda\hat{h}_d - \delta}$ and second derivative $S_{\hat{h}_d,\hat{h}_d} = \Delta''(\hat{h}_d) - \frac{\lambda\eta}{(\lambda\hat{h}_d-\delta)^2}$. The latter is always negative. The former is strictly positive if $\hat{h}_d > \frac{\delta}{\lambda}$, and strictly negative whenever $H < \hat{h}_d \leq \frac{\delta}{\lambda}$. Thus, S can only equal $h_r(0)$ once; indeed, $\hat{h}_d > \frac{\delta}{\lambda}$ iff $h_d(0) > \frac{\delta}{\lambda}$.

A desperate phase requires that profits are positive, which we now verify using the rearranged profit function $\hat{\Pi}(t)$ from Proposition 2. Note that $\hat{\Pi}(T_d) = 0$ by construction. If we take the derivative w.r.t. t, we obtain:

$$\hat{\Pi}'(t) = \left(\frac{x}{\rho} - p_d(t)\right) h_d'(t) - p_d'(t) h_d(t) - \frac{y-x}{\rho} h_r'(t).$$

It is always the case that $\frac{x}{\rho} > p_d(t)$ and $h_d(t) \geq 0$. In a desperate phase, we also have

$h_r'(t) = \eta > 0$. Since $a_d > 0$ by Lemma 3, we know $p_d'(t) = \rho a_d e^{\rho t} > 0$. If $h_d(t) \geq \frac{\delta}{\lambda}$, then $q_d \geq 0$ so $h_d'(t) = -\lambda q_d e^{-\lambda t} \leq 0$. Thus, $\hat{\Pi}'(t) < 0$. At $t = T_d$, profit is zero, so profit must be positive for $t < T_d$ when $h_d(0) \geq \frac{\delta}{\lambda}$.

When $h_d(0) < \frac{\delta}{\lambda}$, it is easier to examine profit in terms of the populations rather than time. After substitution, we obtain:

$$\hat{\Pi}(h_d) = \frac{\lambda h_d \left(c - \frac{c\hat{h}_d + (x-y)\Delta(\hat{h}_d)}{\hat{h}_d}\left(\frac{\delta - \lambda \hat{h}_d}{\delta - \lambda h_d}\right)^{\rho/\lambda}\right) + (y-x)\left(\eta \log\left(\frac{\delta - h_d\lambda}{\delta - \lambda\hat{h}_d}\right) - \lambda\Delta(\hat{h}_d)\right)}{\lambda\rho}.$$

Evaluated at $h_d(0)$, $\hat{\Pi}(h_d) > 0$ is equivalent to $h_r(0) < \frac{c - \rho a_d}{y-x} h_d(0)$, which holds by assumption in Case 3. We then compute the second derivative as:

$$\hat{\Pi}''(h_d) = -\frac{\lambda\eta(y-x)\hat{h}_d + \rho(2\delta + h_d(\rho - \lambda))(c\hat{h}_d - (y-x)\Delta(\hat{h}_d))\left(\frac{\delta - \lambda\hat{h}_d}{\delta - h_d\lambda}\right)^{\rho/\lambda}}{\lambda\rho\hat{h}_d(\delta - h_d\lambda)^2} < 0.$$

The inequality holds because $h_d < \frac{\delta}{\lambda}$ and $c\hat{h}_d > (y-x)\Delta(\hat{h}_d)$ per Lemma 3. Since $\hat{\Pi}(h_d(0)) > 0$ and $\hat{\Pi}(\hat{h}_d) = 0$ while $\hat{\Pi}''(h_d) < 0$ for all $h_d \in (h_d(0), \hat{h}_d)$, profits cannot rise again after falling below zero. Therefore, $\hat{\Pi}(h_d) > 0$ for all $h_d \in (h_d(0), \hat{h}_d)$.

Moving to Case 4, the proof of Case 3 also applies to the desperate phase here, since the assumption $S(\tilde{h}_d, \hat{h}_d) = \frac{c - \rho a_d}{y-x}\tilde{h}_d$ is equivalent to $\hat{\Pi}(\tilde{h}_d) = 0$. We now must verify that relative profits are negative in the initial relaxed phase. This relaxed path follows a straight line segment from $(h_d(0), h_r(0))$ to $\left(\tilde{h}_d, \frac{\eta}{\lambda} + \frac{(\lambda\tilde{h}_d - \delta)q_r}{\lambda q_d}\right)$. The assumption $S(\tilde{h}_d, \hat{h}_d) = \frac{\eta}{\lambda} + \frac{(\lambda\tilde{h}_d - \delta)q_r}{\lambda q_d}$ ensures that the desperate phase continues from the latter point.

To show that profits are negative during the relaxed phase, we use $\tilde{\Pi}(t)$ from before. Recall that $\tilde{\Pi}'(t) = -(y-x) \cdot \frac{\eta h_d(t) - \delta h_r(t)}{h_d(t)^2} - \rho a_r(\rho + \lambda)e^{(\rho+\lambda)t}$. Here, $\eta h_d(t) - \delta h_r(t) = (\eta h_d(0) - \delta h_r(0))e^{\lambda t}$ by substitution. However, because $\beta\delta > \eta$, note that:

$$\begin{aligned}
0 &< \rho(\beta\delta - \eta)\left(\hat{h}_d e^{\rho T_d} + \frac{\delta F(\cdot)}{2\lambda}\right) + \beta\delta\lambda\hat{h}_d\left(e^{\rho T_d} - 1\right) \iff \\
\frac{\eta}{\delta} &< \frac{c - \frac{e^{-\rho T_d}\left(c\hat{h}_d - (y-x)\left(\beta\hat{h}_d + \frac{\beta\delta - \eta}{2\lambda}F(\cdot)\right)\right)}{\hat{h}_d}}{y-x} \iff \\
\frac{\eta}{\delta} &< \frac{c - \frac{e^{-\rho T_d}(c\hat{h}_d - (y-x)\Delta(\hat{h}_d))}{\hat{h}_d}}{y-x} \iff \frac{\eta}{\delta} < \frac{c - a_d}{y-x}.
\end{aligned}$$

Therefore, if $h_r(0) \leq \frac{\eta}{\delta}h_d(0)$, then $h_r(0) < \frac{c - a_d}{y-x}h_d(0)$, so that Case 3 would apply instead of Case 4. Thus $\delta h_r(0) > \eta h_d(0)$.

Moreover, substituting into the definition of a_r:

$$\begin{aligned} a_r &= -\frac{e^{-(\lambda+\rho)T_b}\left(\lambda c\hat{h}_d - (\lambda+\rho)\left(c\hat{h}_d - (y-x)\left(\beta\hat{h}_d + \frac{\beta\delta-\eta}{2\lambda}F(\cdot)\right)\right)e^{\rho(T_b-T_d)}\right)}{\rho(\lambda+\rho)\hat{h}_d} \\ &= -\frac{e^{-(\lambda+\rho)T_b}\left(\lambda c\hat{h}_d\left(1-e^{\rho(T_b-T_d)}\right) + \frac{(\lambda+\rho)(y-x)(\beta\delta-\eta)}{2\lambda}F(\cdot)e^{\rho(T_b-T_d)}\right)}{\rho(\lambda+\rho)\hat{h}_d} < 0. \end{aligned}$$

Hence $\tilde{\Pi}'(t) > 0$ for $t < T_b$. Since $\Pi(T_b) = 0$, therefore $\Pi(t) < 0$ for all $t < T_b$. $\square$