Coaches on the Hot Seat:

A Technical Appendix

Introduction

This technical appendix provides the comparative statics and other analysis used in the proof of main proposition of our paper. Notation is identical to that in the paper.

Bayesian updating

The probability of obtaining w wins is binomially distributed:

$$\text{Binomial}[n, w]\, (1 - p_i)^{n-w}\, p_i^{\,w}$$

Hence, Bayesian updating of prior r results in posterior $\hat{r}$:

$$\hat{r} == \text{FullSimplify}\Big[(1-\gamma)\,\frac{r\,\text{Binomial}[n, w]\,(1-p_g)^{n-w}\,p_g^{\,w}}{r\,\text{Binomial}[n, w]\,(1-p_g)^{n-w}\,p_g^{\,w} + (1-r)\,\text{Binomial}[n, w]\,(1-p_b)^{n-w}\,p_b^{\,w}} + \gamma\,\frac{(1-r)\,\text{Binomial}[n, w]\,(1-p_b)^{n-w}\,p_b^{\,w}}{r\,\text{Binomial}[n, w]\,(1-p_g)^{n-w}\,p_g^{\,w} + (1-r)\,\text{Binomial}[n, w]\,(1-p_b)^{n-w}\,p_b^{\,w}}\Big]$$

$$\hat{r} == \frac{(-1+r)\,\gamma\,(1-p_b)^n\,p_b^w\,(1-p_g)^w + r\,(-1+\gamma)\,(1-p_b)^w\,(1-p_g)^n\,p_g^w}{(-1+r)\,(1-p_b)^n\,p_b^w\,(1-p_g)^w - r\,(1-p_b)^w\,(1-p_g)^n\,p_g^w}$$

$$\hat{r} == \gamma + \text{FullSimplify}\Big[\big((-1+r)\,\gamma\,(1-p_b)^n\,p_b^w\,(1-p_g)^w + r\,(-1+\gamma)\,(1-p_b)^w\,(1-p_g)^n\,p_g^w - \gamma\,\big((-1+r)\,(1-p_b)^n\,p_b^w\,(1-p_g)^w - r\,(1-p_b)^w\,(1-p_g)^n\,p_g^w\big)\big)\,/\,\big((-1+r)\,(1-p_b)^n\,p_b^w\,(1-p_g)^w - r\,(1-p_b)^w\,(1-p_g)^n\,p_g^w\big)\Big]$$

$$\hat{r} == \gamma + \frac{-1+2\gamma}{-1 + \frac{(-1+r)\,(1-p_b)^{n-w}\,p_b^w\,(1-p_g)^{-n+w}\,p_g^{-w}}{r}}$$

$$\hat{r} == \gamma + \frac{r\,(1-2\gamma)}{r + (1-r)\,\left(\frac{p_b}{p_g}\right)^w \left(\frac{1-p_b}{1-p_g}\right)^{n-w}}$$

The firing threshold is given by:

$$\text{FullSimplify}\Big[\text{Solve}\Big[v\,n\,\big((r_0\,p_g + (1-r_0)\,p_b) - (\bar{r}\,p_g + (1-\bar{r})\,p_b)\big) == T, \bar{r}\Big]\Big]$$

$$\left\{\left\{\bar{r} \to \frac{T}{n\,v\,p_b - n\,v\,p_g} + r_0\right\}\right\}$$

Claim 1

The comparative statics on $\hat{r}$ with respect to w is:

$$\text{FullSimplify}\left[D\left[\gamma+\frac{r\,(1-2\gamma)}{r+(1-r)\left(\frac{p_b}{p_g}\right)^{w}\left(\frac{1-p_b}{1-p_g}\right)^{n-w}},\,w\right]\right]$$

$$\frac{(-1+r)\,r\,(-1+2\gamma)\left(\text{Log}\left[\frac{-1+p_b}{-1+p_g}\right]-\text{Log}\left[\frac{p_b}{p_g}\right]\right)\left(\frac{-1+p_b}{-1+p_g}\right)^{n+w}\left(\frac{p_b}{p_g}\right)^{w}}{\left(r\left(\frac{-1+p_b}{-1+p_g}\right)^{w}-(-1+r)\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}\right)^{2}}$$

Note that the denominator must be positive, as will the last two terms of the numerator. Since $0 < r < 1$ and $0 \le \gamma < 1/2$, the first three terms are also positive. Finally, since $p_b < p_g$, then $\text{Log}\left[\frac{p_b}{p_g}\right] < 0$ and $\text{Log}\left[\frac{1-p_b}{1-p_g}\right] > 0$. Thus, more wins always gives a higher posterior.

The comparative static of $\overline{r}$ with respect to w is 0:

$$\text{FullSimplify}\left[D\left[\frac{T}{n\,v\,p_b-n\,v\,p_g}+r0,\,w\right]\right]$$

0

Claim 2

The comparative statics on $\hat{r}$ with respect to r is:

$$\text{FullSimplify}\left[D\left[\gamma+\frac{r\,(1-2\gamma)}{r+(1-r)\left(\frac{p_b}{p_g}\right)^{w}\left(\frac{1-p_b}{1-p_g}\right)^{n-w}},\,r\right]\right]$$

$$\frac{(1-2\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{n+w}\left(\frac{p_b}{p_g}\right)^{w}}{\left(r\left(\frac{-1+p_b}{-1+p_g}\right)^{w}-(-1+r)\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}\right)^{2}}$$

Note that the denominator must be positive, as will the last two terms of the numerator. Since $0 \le \gamma < 1/2$, the first term is also positive. Thus, a stronger past record (encapsulated in r) always gives a higher posterior.

The comparative static of $\overline{r}$ with respect to r is 0.

$$\text{FullSimplify}\left[D\left[\frac{T}{n\,v\,p_b-n\,v\,p_g}+r0,\,r\right]\right]$$

0

To demonstrate that prior seasons have less weight than current seasons, let w indicate season 1 wins, while x indicates season 2 wins. Thus, after these two season, the posterior will be:

$$\text{FullSimplify}\left[\gamma+\frac{r1\,(1-2\gamma)}{r1+(1-r1)\left(\frac{p_b}{p_g}\right)^{x}\left(\frac{1-p_b}{1-p_g}\right)^{n-x}}\ \text{/.}\ r1\to\gamma+\frac{r\,(1-2\gamma)}{r+(1-r)\left(\frac{p_b}{p_g}\right)^{w}\left(\frac{1-p_b}{1-p_g}\right)^{n-w}}\right]$$

$$\gamma+\frac{-1+2\gamma}{-1+\frac{\left(r\,\gamma\left(\frac{-1+p_b}{-1+p_g}\right)^{w}+(-1+r)\,(-1+\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}\right)\left(\frac{-1+p_b}{-1+p_g}\right)^{n-x}\left(\frac{p_b}{p_g}\right)^{x}}{r\,(-1+\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{w}+(-1+r)\,\gamma\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}}}$$

Now we compare the marginal effect of increasing x versus increasing w:

$$\texttt{FullSimplify}\left[\texttt{D}\left[\gamma+\frac{-1+2\gamma}{-1+\frac{\left(r\gamma\left(\frac{-1+p_b}{-1+p_g}\right)^{w}+(-1+r)(-1+\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}\right)\left(\frac{-1+p_b}{-1+p_g}\right)^{n-x}\left(\frac{p_b}{p_g}\right)^{x}}{r(-1+\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{w}+(-1+r)\gamma\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}}},\ w\right]<\right.$$

$$\left.\texttt{D}\left[\gamma+\frac{-1+2\gamma}{-1+\frac{\left(r\gamma\left(\frac{-1+p_b}{-1+p_g}\right)^{w}+(-1+r)(-1+\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}\right)\left(\frac{-1+p_b}{-1+p_g}\right)^{n-x}\left(\frac{p_b}{p_g}\right)^{x}}{r(-1+\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{w}+(-1+r)\gamma\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}}},\ x\right]\right]$$

$$\gamma(-1+2\gamma)\left(\mathrm{Log}\left[\frac{-1+p_b}{-1+p_g}\right]-\mathrm{Log}\left[\frac{p_b}{p_g}\right]\right)$$
$$\left(r^2(-1+\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{2w}+2(-1+r)r\gamma\left(\frac{-1+p_b}{-1+p_g}\right)^{n+w}\left(\frac{p_b}{p_g}\right)^{w}+(-1+r)^2(-1+\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{2n}\left(\frac{p_b}{p_g}\right)^{2w}\right)$$
$$\left(r\left(\gamma\left(\left(\left(\frac{-1+p_b}{-1+p_g}\right)^{w}+\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}\right)\left(-\left(\frac{-1+p_b}{-1+p_g}\right)^{x}+\left(\frac{-1+p_b}{-1+p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{x}\right)\right)+\right.\right.$$
$$\left.\left(\frac{-1+p_b}{-1+p_g}\right)^{w+x}-\left(\frac{-1+p_b}{-1+p_g}\right)^{2n}\left(\frac{p_b}{p_g}\right)^{w+x}\right)+\gamma\left(\frac{-1+p_b}{-1+p_g}\right)^{n+x}\left(\frac{p_b}{p_g}\right)^{w}+$$
$$\left.\left(\frac{-1+p_b}{-1+p_g}\right)^{2n}\left(\frac{p_b}{p_g}\right)^{w+x}-\gamma\left(\frac{-1+p_b}{-1+p_g}\right)^{2n}\left(\frac{p_b}{p_g}\right)^{w+x}\right)^{2}\left(\frac{-1+p_b}{-1+p_g}\right)^{n+x}\left(\frac{p_b}{p_g}\right)^{x}>0$$

$$\gamma(1-2\gamma)\left(\mathrm{Log}\left[\frac{1-p_b}{1-p_g}\right]-\mathrm{Log}\left[\frac{p_b}{p_g}\right]\right)$$
$$\left(r^2(1-\gamma)\left(\frac{1-p_b}{1-p_g}\right)^{2w}+2(1-r)r\gamma\left(\frac{1-p_b}{1-p_g}\right)^{n+w}\left(\frac{p_b}{p_g}\right)^{w}+(1-r)^2(1-\gamma)\left(\frac{1-p_b}{1-p_g}\right)^{2n}\left(\frac{p_b}{p_g}\right)^{2w}\right)$$
$$\left(r\left(\gamma\left(\left(\left(\frac{1-p_b}{1-p_g}\right)^{w}+\left(\frac{1-p_b}{1-p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{w}\right)\left(-\left(\frac{1-p_b}{1-p_g}\right)^{x}+\left(\frac{1-p_b}{1-p_g}\right)^{n}\left(\frac{p_b}{p_g}\right)^{x}\right)\right)+\right.\right.$$
$$\left.\left(\frac{1-p_b}{1-p_g}\right)^{w+x}-\left(\frac{1-p_b}{1-p_g}\right)^{2n}\left(\frac{p_b}{p_g}\right)^{w+x}\right)+\gamma\left(\frac{1-p_b}{1-p_g}\right)^{n+x}\left(\frac{p_b}{p_g}\right)^{w}+$$
$$\left.\left(\frac{1-p_b}{1-p_g}\right)^{2n}\left(\frac{p_b}{p_g}\right)^{w+x}-\gamma\left(\frac{1-p_b}{1-p_g}\right)^{2n}\left(\frac{p_b}{p_g}\right)^{w+x}\right)^{2}\left(\frac{1-p_b}{1-p_g}\right)^{n+x}\left(\frac{p_b}{p_g}\right)^{x}>0$$

The last three terms are positive, as are the first three terms (strictly so if $0<\gamma<1/2$).

$$\left(r^2(1-\gamma)\left(\frac{1-p_b}{1-p_g}\right)^{2w}+2(1-r)r\gamma\left(\frac{1-p_b}{1-p_g}\right)^{n+w}\left(\frac{p_b}{p_g}\right)^{w}+(1-r)^2(1-\gamma)\left(\frac{1-p_b}{1-p_g}\right)^{2n}\left(\frac{p_b}{p_g}\right)^{2w}\right)>0$$

The remaining term is also necessarily positive since $0<r<1$ and $0\le\gamma<1/2$.

Claim 3

The comparative statics on $\hat{r}$ with respect to p_g is:

$$\text{FullSimplify}\left[D\left[\gamma + \frac{r\,(1-2\gamma)}{r + (1-r)\left(\frac{p_b}{p_g}\right)^w \left(\frac{1-p_b}{1-p_g}\right)^{n-w}}, p_g\right]\right]$$

$$\frac{(-1+r)\,r\,(-1+2\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{n+w}\left(\frac{p_b}{p_g}\right)^w (-w+n\,p_g)}{\left(r\left(\frac{-1+p_b}{-1+p_g}\right)^w - (-1+r)\left(\frac{-1+p_b}{-1+p_g}\right)^n \left(\frac{p_b}{p_g}\right)^w\right)^2 (-1+p_g)\,p_g}$$

$$\frac{(1-r)\,r\,(1-2\gamma)\left(\frac{1-p_b}{1-p_g}\right)^{n+w}\left(\frac{p_b}{p_g}\right)^w (w-n\,p_g)}{\left(r\left(\frac{1-p_b}{1-p_g}\right)^w - (-1+r)\left(\frac{1-p_b}{1-p_g}\right)^n \left(\frac{p_b}{p_g}\right)^w\right)^2 (1-p_g)\,p_g}$$

All of these terms are always positive except the last in the numerator. Thus, the posterior is higher after an increase in p_g iff $w > n\,p_g$

The comparative static of $\overline{r}$ with respect to r is positive.

$$\text{FullSimplify}\left[D\left[\frac{T}{n\,v\,p_b - n\,v\,p_g} + r0,\ p_g\right]\right]$$

$$\frac{n\,T\,v}{(n\,v\,p_b - n\,v\,p_g)^2}$$

If instead p_g and p_b increase equally, the threshold is unaffected. The posterior changes as follows:

$$\text{FullSimplify}\left[D\left[\gamma + \frac{r\,(1-2\gamma)}{r + (1-r)\left(\frac{p_b}{p_g}\right)^w \left(\frac{1-p_b}{1-p_g}\right)^{n-w}}, p_g\right] + D\left[\gamma + \frac{r\,(1-2\gamma)}{r + (1-r)\left(\frac{p_b}{p_g}\right)^w \left(\frac{1-p_b}{1-p_g}\right)^{n-w}}, p_b\right]\right]$$

$$\left((-1+r)\,r\,(-1+2\gamma)\,(p_b-p_g)\left(\frac{-1+p_b}{-1+p_g}\right)^{-1+n+w}\left(\frac{p_b}{p_g}\right)^{-1+w}(w - w\,p_g + p_b\,(-w+n\,p_g))\right)\Big/ \left(\left(r\left(\frac{-1+p_b}{-1+p_g}\right)^w - (-1+r)\left(\frac{-1+p_b}{-1+p_g}\right)^n\left(\frac{p_b}{p_g}\right)^w\right)^2 (-1+p_g)^2\,p_g^2\right)$$

$$\left((1-r)\,r\,(1-2\gamma)\,(p_g-p_b)\left(\frac{1-p_b}{1-p_g}\right)^{-1+n+w}\left(\frac{p_b}{p_g}\right)^{-1+w}(w\,(p_g+p_b-1) - n\,p_g\,p_b)\right)\Big/ \left(\left(r\left(\frac{1-p_b}{1-p_g}\right)^w - (-1+r)\left(\frac{1-p_b}{1-p_g}\right)^n\left(\frac{p_b}{p_g}\right)^w\right)^2 (1-p_g)^2\,p_g^2\right)$$

All terms are positive except the last in the numerator. If we assume that $p_g + p_b \leq 1$, then that term is always negative.

Claim 4

The threshold must lie below $r_0$because $p_g > p_b$:

$$\overline{r} == r_0 - \frac{T}{n\,v\,(p_g - p_b)}$$

Claim 6

Whatever the firing threshold is, if you survived your preiovus season with some posterior r, we know you will not be fired if your next season posterior is above r. The posterior goes down, relative to to the prior, iff:

$$\gamma + \frac{r\,(1-2\gamma)}{r + (1-r)\left(\frac{p_b}{p_g}\right)^{w}\left(\frac{1-p_b}{1-p_g}\right)^{n-w}} > r$$

$$\text{FullSimplify}\left[r\,(1-2\gamma) > (r-\gamma)\left(r + (1-r)\left(\frac{p_b}{p_g}\right)^{w}\left(\frac{1-p_b}{1-p_g}\right)^{n-w}\right)\right]$$

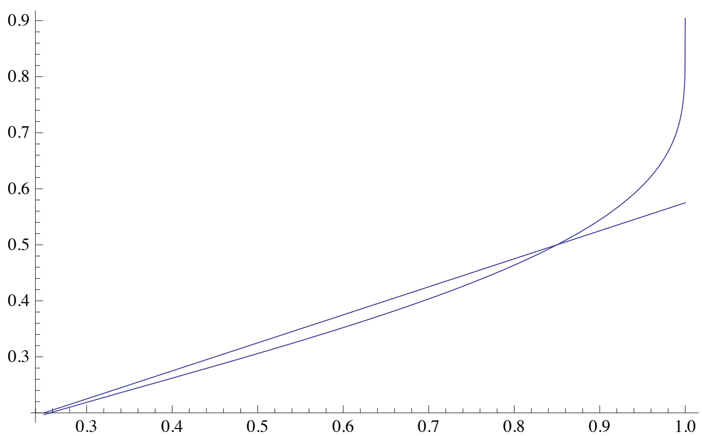
$$r + (-1+r)\,(r-\gamma)\left(\frac{-1+p_b}{-1+p_g}\right)^{n-w}\left(\frac{p_b}{p_g}\right)^{w} > r\,(r+\gamma)$$

$$(n-w)\,\text{Log}\left[\frac{1-p_b}{1-p_g}\right] + w\,\text{Log}\left[\frac{p_b}{p_g}\right] < \text{Log}\left[\frac{r\,(1-r+\gamma)}{(1-r)\,(r-\gamma)}\right]$$

$$w > \left(\text{Log}\left[\frac{r\,(1-r+\gamma)}{(1-r)\,(r-\gamma)}\right] + n\,\text{Log}\left[\frac{1-p_g}{1-p_b}\right]\right) \Big/ \text{Log}\left[\frac{p_b}{p_g}\,\frac{1-p_g}{1-p_b}\right]$$

If $\gamma = 0$, the first term in the numerator disappears. The remaining terms are closely approximated by $(p_g + p_b)/2$ so long as neither p_g nor p_b are too extreme:

$$\text{Plot}\left[\left\{\text{Log}\left[\frac{1-p_g}{1-p_b}\right] \Big/ \text{Log}\left[\frac{p_b}{p_g}\,\frac{1-p_g}{1-p_b}\right],\ (p_g+p_b)/2\right\} /.\ p_b \to 0.15,\ \{p_g, 0.25, 1\}\right]$$



For values near $\gamma = 0$, and increase in γ will decrease this threshold.

$$\text{FullSimplify}\left[D\left[\left(\text{Log}\left[\frac{r\,(1-r+\gamma)}{(1-r)\,(r-\gamma)}\right] + n\,\text{Log}\left[\frac{1-p_g}{1-p_b}\right]\right) \Big/ \text{Log}\left[\frac{p_b}{p_g}\,\frac{1-p_g}{1-p_b}\right], \gamma\right] /.\ \gamma \to 0\right]$$

$$\frac{1}{(r - r^2)\,\text{Log}\left[\frac{p_b\,(-1+p_g)}{(-1+p_b)\,p_g}\right]}$$

The log term is always negative, since $p_g > p_b$.