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The Journal of Business, Vol. 60, No. 3. (Jul., 1987), pp. 401-424.

Stable URL:
http://links.jstor.org/sici?sici=0021-9398%28198707%2960%3A3%3C401%3AAGDFDS%3E2.0.CO%3B2-C

The Journal of Business is currently published by The University of Chicago Press.

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A General Distribution for Describing Security Price Returns

This paper introduces a generalized distribution, called the GB2 distribution, for describing security returns. The distribution is extremely flexible, containing a large number of well-known distributions, such as the lognormal, log-t, and log-Cauchy distributions, as special or limiting cases and allowing large, even infinite higher moments. This flexibility allows a direct representation of different degrees of fat tails in the distribution. The properties of the GB2 make it useful in empirical estimation of security returns and in facilitating the development of option pricing models and other models that depend on the specification and mathematical manipulation of distributions.

I. Introduction

This paper introduces a generalized distribution for describing security returns. The distribution has the feature of being extremely flexible, and it includes a large number of well-known distributions, such as the lognormal, log-t, and log-Cauchy distributions, as special or limiting cases. Distributions with large, even infinite higher moments can be specified by the choice of parameters. This flexibility allows a direct representation of different degrees of fat tails in the distribution. The generalized distribution also has a natural relation to much of the literature on mixed distributions since a wide range of mixed distributions can be described as special cases of this distribution.

There are two common approaches to the study of the distribution of security returns in the finance literature. The first begins by describing the process that gives rise to the returns, and the second begins by seeking to represent in a usable form a distribution function that empirically fits the observed return distribution. Much of the literature that relies on mixed distributions takes the first approach as its starting point and in doing so emphasizes the market process and the relation between various market variables, such as price variability and trading volume. A number of these papers lead to well-defined distributions. Others, which examine the trading process

(*Journal of Business*, 1987, vol. 60, no. 3)

0021-9398/87/6003-0005$01.50

in greater detail, such as those of Epps and Epps (1976), Oldfield, Rogalski, and Jarrow (1977), and Tauchen and Pitts (1983), lead to distributions that cannot be represented in explicit form or are difficult to specify and use in application.

The second approach serves as the starting point for a line of research that has its roots in the work of Fama (1963, 1965) and Mandelbrot (1963). This work begins with the empirical observation that stock returns are more peaked and have thicker tails than the lognormal and then finds a distribution function that fits this observation. One such set of distributions is characterized by a set of symmetric-stable distributions with characteristic exponents between one and two. These distributions are chosen both because of their fit to the observed distributions and because they have the attractive property of closure under multiplication. That is, the product of security returns will retain the same distributional form as for individual returns. There appears to have been little if any work to link this set of distributions to the actual mechanism of security trading. In this respect, these distributions remain only an empirical description of the fitted distributions.[1]

The generalized distribution we present in this paper has the advantage of being easily interpreted as a mixed distribution and has an easily expressible density function that makes it amenable to both empirical and theoretical work in which the density must be expressed explicitly.

In addition to presenting the generalized distribution function, this paper also introduces a valuable statistical technique called bootstrapping into the finance literature. While the field of finance is well endowed with data, it is nonetheless difficult to generate a large sample of return distribution series, particularly for longer time periods. For example, given the large number of samples necessary to evaluate a return distribution, it is difficult to measure the return distribution of annual, or even monthly, returns. It would take 10 years of data to generate 120 nonoverlapping observations of monthly returns, a modest sample for many statistical estimation purposes, and the changing character of many firms would make it difficult to argue that samples drawn over so wide a time period are representative of the same population.

Bootstrapping provides a means of generating a large number of samples for statistical hypothesis testing from a smaller sample population. Bootstrapping essentially involves replicating the initial sample many times and then drawing with replacement from this large sample set. We rely on this technique in order to evaluate the distribution of

1. Since these distributions must generally be expressed in terms of their characteristic function, they also face limitations for many applications, particularly those related to the development of option pricing models. Explicit expressions for the density function are reportedly known for only two special cases (Officer 1972).

return as the time period of the return goes from daily to annual. The same underlying sample set is used for all lengths of returns.

The paper is organized as follows. The next section of the paper introduces the generalized distribution and describes a number of its properties. Section III demonstrates the application of the distribution in addressing problems that arise because of nonstationarity. Nonstationarity arises from two sources: the parameter values may themselves be stochastic; and the distribution may not have closure under multiplication, so that returns of differing lengths will have different distributions. The first source of nonstationarity gives rise to mixed distributions. The second source is less commonly addressed.[2] Indeed, the stable Paretian family of distributions was selected precisely because it had the desired closure property. Section IV presents the empirical results of comparisons of the generalized and the lognormal distributions. Since the lognormal is a limiting case of the generalized distribution, the direction of the comparative results is clear. The generalized distribution must perform at least as well as the lognormal or as the log t and other special cases. However, the magnitude of improvement sheds some light on the descriptive power of the generalized distribution and presents some new insights into the areas of failure of the lognormal in describing security returns. Section V then concludes the paper.

II. A Generalized Distribution Function

We term the generalized distribution that forms the basis for the analysis of this paper the generalized beta of the second kind (GB2). The GB2 is defined as follows:

$$\mathrm{GB2}(y; a, b, p, q) = \frac{|a|y^{ap-1}}{b^{ap}B(p, q)[1 + (y/b)^a]^{p+q}}, \tag{1a}$$

$y > 0$, and

$$\mathrm{GB2}(y; a, b, p, q) = 0 \tag{1b}$$

otherwise. The distribution function for the GB2 is

$$\frac{\left[\frac{(y/b)^a}{1 + (y/b)^a}\right]^p}{pB(p, q)}, \quad {}_2F_1\left[\begin{matrix} p, 1 - q; \ \frac{(y/b)^a}{1 + (y/b)^a} \\ p + 1; \end{matrix}\right]. \tag{2}$$

The associated expression for the hth moment is

$$\frac{b^h B(p + h/a, q - h/a)}{B(p, q)}. \tag{3}$$

2. Some consideration of the closure property, stability over additivity in prices, or stability in multiplication in returns is presented in Fielitz and Rozelle (1983).

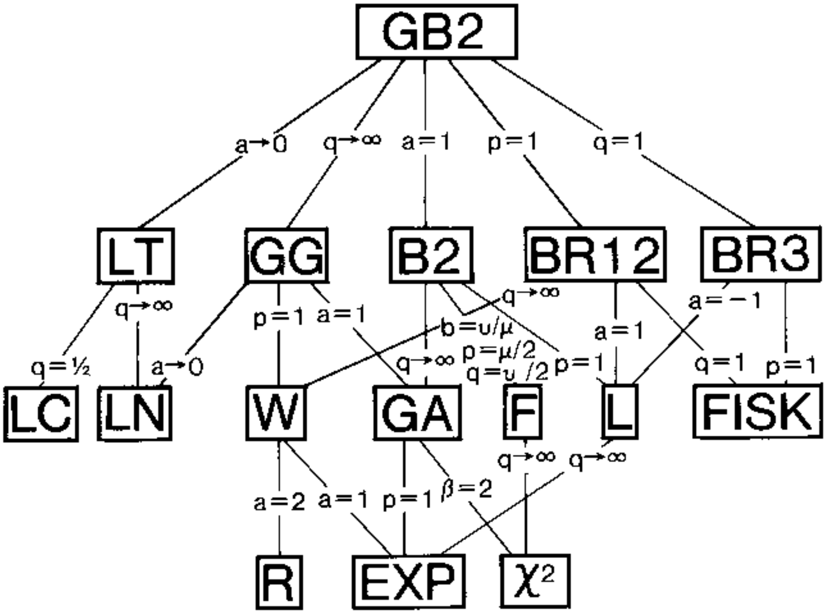

FIG. 1.—Distribution family tree. LT = log t; B2 = beta of the second kind; BR12 = Singh-Maddala or Burr type 12; BR3 = Burr type 3; LC = log Cauchy; LN = lognormal; W = Weibull; GA = gamma; L = Lomax; R = Rayleigh; EXP = exponential.

This representation for the distribution and moments is from McDonald (1984). The ${}_2F_1[\quad]$ and B( ) in (2) and (3) denote the hypergeometric series and the beta function. A distribution of a form similar to (1) for positive values of the parameter a has been called the generalized F distribution by Kalbfleisch and Prentice (1980) and for nonzero thresholds the Feller-Pareto distribution by Arnold (1983).

The relation of the GB2 to other distributions is shown in figure 1. From this figure, we can see that the GB2 includes the generalized gamma (GG) as a limiting case:

$$\mathrm{GG}(y; a, \beta, p) = \lim_{q \to \infty} \mathrm{GB2}(y; a, \beta q^{1/a}, p, q). \quad (4)$$

Further limits applied to the GG lead to the lognormal density as a special limiting case of the GB2:

$$\mathrm{LN}(y; \mu, \sigma) = \lim_{a \to 0} \mathrm{GG}[y; a, \beta = (\sigma^2 a^2)^{1/a}, p = (a\mu + 1)/\beta^a]. \quad (5)$$

In addition to these two distributions, figure 1 indicates a wide range of well-known distributions that can be expressed as limiting and special cases of the GB2. These cases include the log t and the lognormal, two cases of particular interest in the study of the distribution of security returns; we will consider these in more detail below. They also include other distributions of interest for statistical hypothesis testing, such as the chi square, F, exponential, and gamma; distributions for measuring income distributions, such as the Singh-Maddala or Burr type 12, Fisk, and GG; and distributions from engineering and reliability measurement applications, such as the Burr type 3, Weibull, Rayleigh, and Lomax.[3]

A number of further limiting distributions occur as the value of p approaches infinity. The limiting case of the GB2 as p goes to infinity is the inverse generalized gamma (IGG).[4] This adds an additional dimension to figure 1 since the inverse gamma, inverse Weibull, inverse exponential, and inverse Rayleigh will all be further restricted cases of this distribution.

The GB2 is a function of four parameters a, b, p, and q. These parameters work interactively in determining the shape of the distribution. A characterization of the parameters on the distribution can be made through a consideration of the moments of the distribution. However, a few general observations can be made to give more intuitive feel for the effect of the parameters on the distribution.

The b parameter is a scale parameter, which stretches out or shrinks the distribution. For large values of the a parameter, the scale parameter has a direct relation to the mean of the distribution. For a large value of the a parameter, the b will approximate the mean; a doubling of b will move the mean 100% to the right. This is evident from inspection of the expression for the mean:

$$E(y) = [bB(p + 1/a, q - 1/a)]/B(p, q). \tag{6}$$

The effect of the b parameter is also evident from the density function for the GB2. A change in the value of b will have an obvious effect on the height of the density. Figure 2 shows the effect of a change in b on the distribution.

The a, p, and q parameters are shape parameters. The a parameter determines the "speed" with which the tails of the density function approach the X-axis. A higher value of the a parameter implies a quicker approach to the axis, as is shown in figure 3.[5] The product aq has a direct effect on the fatness of the distribution. In particular, no moments of order equal to or higher than aq will exist. This thus presents

3. Other distributions, such as the normal, student t, and Cauchy, can be generated by reflecting a GB2 distribution about the origin.

4. The inverse distribution corresponds to the distribution of the reciprocal variable. For example, if y is distributed GG, then $1/y$ is distributed IGG.

5. As can be seen from inspection of eq. (6), a change in a will also have some effect in the location of the distribution.

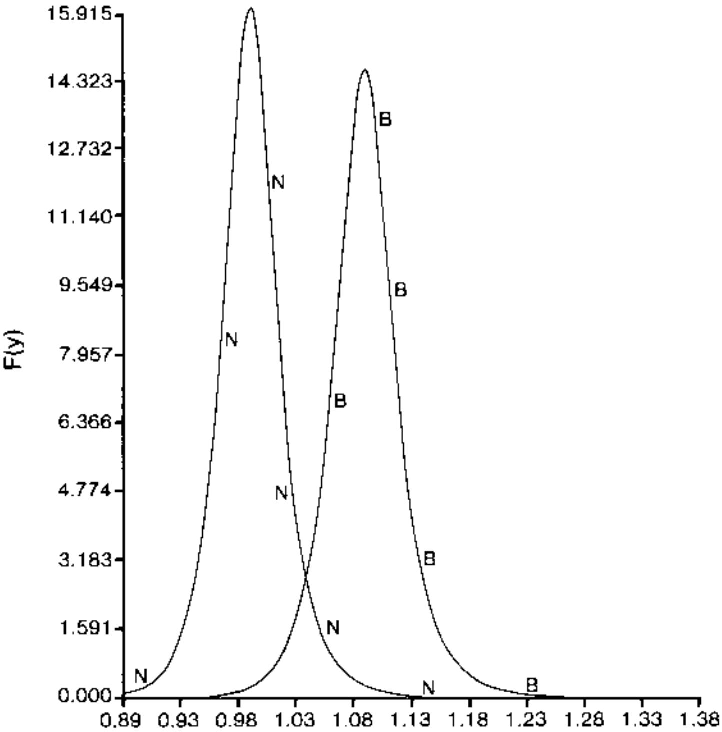

FIG. 2.—GB2: A change in b, from 1.0 to 1.1 (a = 100, p = .5, q = .5). B = change in b; N = no change.

a natural tool for constructing leptokurtotic distributions that seem to typify stock returns. The family of distributions with infinite kurtosis will consist of the GB2 restricted to have $aq < 4$, while the family of infinite-variance distributions will have $aq < 2$. Equation (3) also shows that the p and q parameters will dictate the skewness of the distribution. The GB2 distribution, unlike the lognormal, has the flexibility to allow either positive or negative skewness. Figures 4 and 5 illustrate the effect of changes in p and q on skewness. As is evident from the figures, an increase in p has an effect that is the opposite of that of q.

III. The Use of the GB2 in Modeling Nonstationary Distributions

The distribution of stock returns may have nonstationarity induced from two sources. First, the functional form of the distribution may be

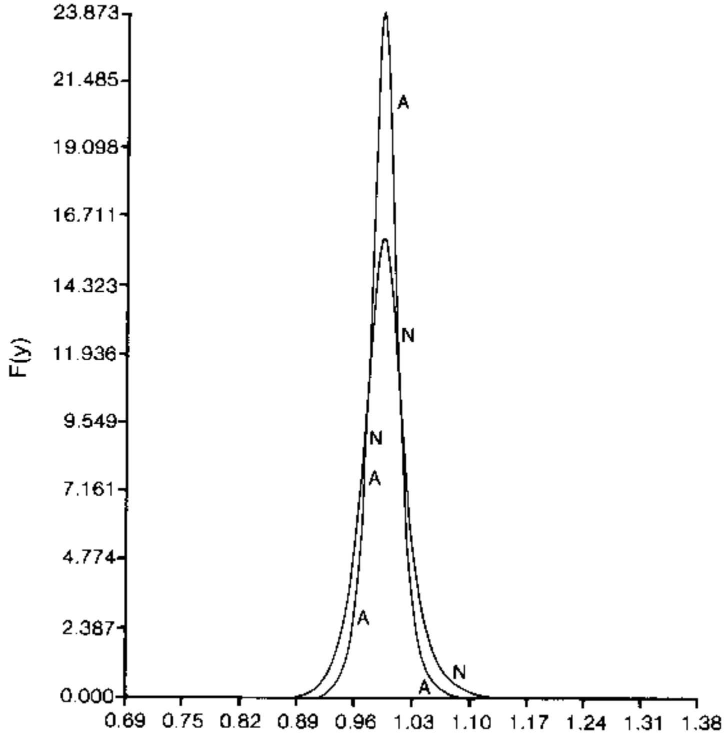

FIG. 3.—GB2: A change in a, from 100 to 150 ($b = 1.0, p = .5, q = .5$). A = change in a; N = no change.

stable, but the parameter values may not be stationary. In particular, stochastic parameter values will lead to a mixed distribution. Second, the distribution may not have closure under multiplication. The distribution for returns for one time length will then vary in functional form from the distribution for another time length. Daily returns need not follow the same functional distribution as monthly returns, for example. In this section we will consider the ability of the GB2 to model these types of nonstationarity. First, we will show the properties of the GB2 distribution for describing mixed distributions, and then we will derive the distribution that results from the product of two GB2-distributed random variables. This product will itself result in a GB2 distribution under certain restrictions, leading to distributional stationarity.

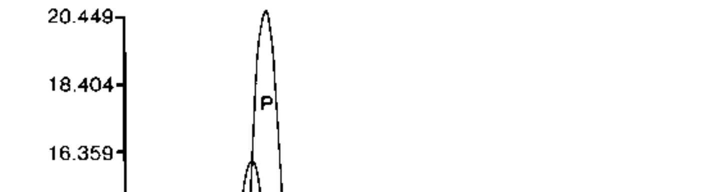

FIG. 4.—GB2: A change in p, from .5 to 1.5 (a = 100, b = 1.0, q = .5). P = change in p; N = no change.

The Use of the GB2 in Generating Mixture Distributions

A widely studied hypothesis of security price distributions is that the security prices involve a mixture of distributions. The mixture of a lognormal distribution of prices with a distribution governing the variance of prices has been shown to lead to the observed degree of leptokurtosis in the distribution of stock price returns. Papers by Praetz (1972) and by Blattberg and Gonedes (1974) show that, if the variance of the distribution follows an inverted gamma distribution, then the resulting distribution of prices is distributed as a log t. Pearson (1967) attributes a mixture interpretation of the t-distribution to Edgeworth. Other papers have used different mixtures on the lognormal to generate distributions that conform with observed price return data. Press (1967) uses Poisson mixtures of normal distributions of returns, while Kon (1984) uses discrete distributions for the mixing density.

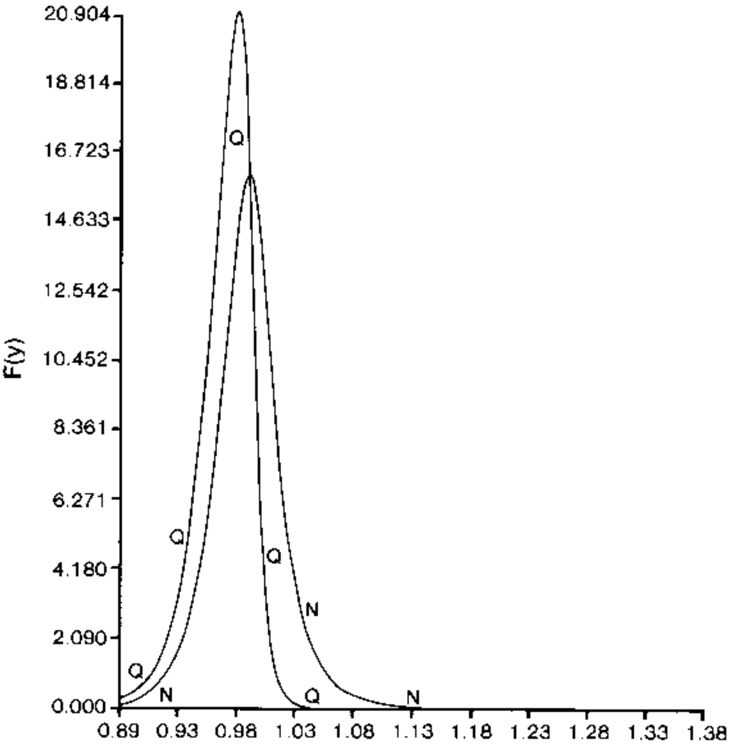

FIG. 5.—GB2: A change in q, from .5 to 1.5 (a = 100, b = 1.0, p = .5). Q = change in q; N = no change.

Clark (1973), Epps and Epps (1976), and Tauchen and Pitts (1983) model the distribution of variance as a function of trading volume or trading activity. The resulting models lead to a mixed distribution in which the population of observed stock prices is heterogenous in a way related to the level of trading activity. The heterogeneity is related to trading activity in a complex fashion that makes it difficult to specify the resulting distribution analytically.[6]

The GB2 distribution has a number of attractive features for studying mixtures of distributions. The GB2 includes as a subset a large number

6. The mixture hypothesis is also treated by Barnea and Downes (1973), Brenner (1974), Oldfield et al. (1977), Westerfield (1977), Dowell and Grube (1978), Castanias (1979), and Perry (1982). The mixture hypothesis can be considered as the chief alternative to the hypothesis that stock prices follow a stable distribution. The latter hypothesis is generally associated with Fama (1963, 1965) and Mandelbrot (1963). The principal model for the stable hypothesis is the stable Paretian distribution. Generally, this distribution can be expressed only through its four-parameter characteristic function.

of mixed distributions, thereby allowing the components of the resulting distribution to be modeled and the resulting distribution to be expressed analytically as a function of these components. Furthermore, the GB2 includes a number of common mixed distributions as special cases. The most notable of these is the log-t distribution that arises from mixing the lognormal with the inverted gamma.

A mixed distribution is generated from two distributions. The first is the *structural distribution* of a variable y for a particular subset of the population. This distribution is assumed to have a density function $f(y; \nu, \Theta)$. The parameter Θ is assumed to vary across subsets of y. It is assumed that the distribution of the variable Θ is specified by a second distribution, a *mixing density* $g(\Theta; \phi)$. The resulting (posterior) distribution for y is the mixture of the density functions $f(y; \nu, \Theta)$ and $g(\Theta; \phi)$. This *observed* or *mixed density* is defined by

$$h(y; \nu, \phi) = \int f(y; \nu, \Theta) g(\Theta; \phi) d\Theta \qquad (7)$$
$$= f(y, \nu, \Theta) \underset{\Theta}{\wedge} g(\Theta, \phi).$$

Figure 1 already demonstrates that a limiting case of the GB2 represents the most widely cited mixed distribution in the finance literature, the log-t (LT) distribution, which is the lognormal (LN) mixed with the inverted gamma (IG):

$$\lim_{a \to 0} \text{GB2}[y; a, b = (\sigma^2 a^2 q)^{1/a}, p = (a\,\mu + 1)/(\sigma^2 a^2), q]$$
$$= \text{LN}(y; \mu, \Theta) \underset{\Theta}{\wedge} \text{IG}(\Theta; \sigma^2 q, q) \qquad (8)$$
$$= \text{LT}(y; \sigma^2, \mu, q).$$

In the context of the GB2 distribution, consider the class of mixed distributions of the form

$$\text{GB2}(y; a, \Theta, p, q) \underset{\Theta}{\wedge} \text{GB2}(\Theta; a, b, \Theta_3, \Theta_4). \qquad (9)$$

The first distribution is the structural distribution, and the second distribution is the mixing distribution. The mixing parameter in this case of mixed distributions is the scale parameter. (Note that the mixing of the lognormal and the gamma to create the log t uses the scale parameter of the lognormal, σ^2, as the mixing parameter.) The density function for the resulting distribution is[7]

$$\frac{|a| B(q + \Theta_3, p + \Theta_4)(y/b)^{a\Theta_3 - 1}}{b B(p, q) B(\Theta_3, \Theta_4)},$$
$${}_2F_1 \left[\begin{matrix} \Theta_3 + \Theta_4, q + \Theta_3; 1 - (y/b)^a \\ p + q + \Theta_3 + \Theta_4; \end{matrix} \right], \qquad (10)$$

7. This result is shown in McDonald and Butler (in press, app.).

for $0 < y \leq b$, and

$$\frac{|a|B(q + \Theta_3, p + \Theta_4)(b/y)^{a\Theta_4+1}}{bB(p, q)B(\Theta_3,\Theta_4)},$$

$${}_2F_1\left[\begin{matrix}\Theta_3 + \Theta_4, p + \Theta_4; 1 - (b/y)^a \\ p + q + \Theta_3 + \Theta_4;\end{matrix}\right],$$

for $y \geq b$.

Given the wide range of distributions contained as special cases of the GB2, it is apparent that there is a broad scope of possible mixed distributions that can be obtained as special or limiting cases of this distribution.

It is interesting to note that the mixture of the GG with the IGG returns the GB2 as the resulting distribution:

$$\text{GG}(y; a, \Theta, p) \underset{\Theta}{\wedge} \text{IGG}(\Theta; a, b, q) = \text{GB2}(y; a, b, p, q). \quad (11)$$

This result provides an interpretation of the GB2 as a mixed distribution. Since the lognormal and the gamma are special or limiting cases of the GG, it suggests the role the GB2 can play as a generalized version of the lognormal-gamma mixture employed by Praetz and others.

Tests for Closure under Multiplication: Deriving the Distribution of the Product of GB2-distributed Random Variables

Closure under multiplication is a desirable property for security distributions. Without this property, a distribution can be considered only for returns of one given time length. If daily returns have one distribution, taking the product of these returns to generate monthly returns may lead to a completely different distribution function. An important property of the lognormal distribution is closure under multiplication. If daily returns are distributed lognormally, the product of these returns will also be distributed lognormally. The nonnormal stable distributions suggested by Mandelbrot (1963) and others as an alternative to the lognormal distribution also have the property that linear combinations of independently and identically distributed variables will be symmetric stable and will preserve the same characteristic exponent. When the log-of-price relatives are typified by this class of distributions, then the returns will retain the property of closure under multiplication as well (see, e.g., Feller 1970, vol. 2; or Teichmoeller 1971).

While such distributional stationarity is appealing, whether in fact such assumptions of stationarity are justified remains an open question. The GB2 distribution can shed some light on this question since it includes distributions such as the lognormal that have closure under multiplication and also admits many cases that do not have such closure.[8]

8. A subset of the GB2 distribution has closure under addition. For example, GG

We can derive the distribution function of the product of two GB2-distributed random variables. Since the product of a set of random variables can be evaluated two at a time, any set of restrictions on the resulting distribution that reduces it to a GB2 distribution will imply closure under multiplication for that restricted case.

To derive the distribution of the product of two independent GB2-distributed variables, let

$$x_i \sim \mathrm{GB2}(x_i; a, b, p, q), \quad i = 1, 2, \tag{12}$$

and define

$$y_1 = x_1 x_2, \tag{13}$$

$$y_2 = x_2. \tag{14}$$

The joint density of x_1 and x_2 is given by

$$g(x_1, x_2) = \frac{|a|^2 x_1^{ap-1} x_2^{ap-1}}{b^{2ap}[B(p,q)]^2[1 + (x_1/b)^a]^{p+q}[1 + (x_2/b)^a]^{p+q}}. \tag{15}$$

The joint density of y_1 and y_2 is then

$$\begin{aligned} h(y_1, y_2) &= g(y_1/y_2, y_2)J(x, y) \\ &= \frac{|a|^2 (y_1/y_2)^{ap-1} y_2^{ap-1}(1/y_2)}{b^{2ap}[B(p, q)]^2[1 + (y_1/by_2)^a]^{p+q}[1 + (y_2/b)^a]^{p+q}} \\ &= \frac{|a|^2 y_1^{ap-1}}{b^{2ap}[B(p, q)]^2} \\ &\quad \times \frac{(y_2/b)^{a(p+q)}(1/y_2)}{[1 + (y_2/b)^a]^{p+q}[(y_2/b)^a + (y_1/b)^a]^{p+q}}. \end{aligned} \tag{16}$$

The function $J(x, y)$ is the Jacobian of the transformation from (x_1, x_2) to (y_1, y_2).

The density of y_1 is obtained by integrating out the y_2 variable. This is facilitated by making the transformation[9]

$$s = (y_2/b)^a. \tag{17}$$

Making this substitution yields

$$h(y_1) = \frac{|a| y_1^{ap-1}}{b^{2ap}[B(p, q)]^2} \int_0^\infty \frac{s^{p+q-1}}{(1 + s)^{p+q}[s + (y_1/b^2)^a]^{p+q}} ds. \tag{18}$$

distributions with the a parameter equal to one and the same b parameter will have closure under addition. If the log-of-price relatives are typified by the GG, these properties will be valuable.

9. Thus $ds = |a|(y_2/b)^a \, dy_2/y_2$, or $dy_2/y_2 = (1/|a|)ds/s$.

The integral can be evaluated to give[10]

$$h(y_1) = \frac{|a|(y_1/b^2)^{ap}B(p+q, p+q)}{y_1[B(p, q)]^2}, \quad {}_2F_1\left[\begin{matrix} p+q, p+q;\ 1-(y_1/b^2)^a \\ 2p+2q; \end{matrix}\right], \tag{19}$$

for $0 < y_1 < b^2$, and

$$h(y_1) = \frac{|a|(y_1/b^2)^{-aq}B(p+q, p+q)}{y_1[B(p, q)]^2}, \quad {}_2F_1\left[\begin{matrix} p+q, p+q;\ 1-(b^2/y_1)^a \\ 2p+2q; \end{matrix}\right],$$

for $y_1 > b^2$.

This density function as stated is clearly not identical in form to the GB2 density function. Therefore, as we already mentioned, the GB2 distribution will not have closure under multiplication without further restrictions. It can be verified that the restrictions imposed by the lognormal, for example, reduce this density function to such a form. Other restrictions must be tested against this density function on a case-by-case basis.

IV. Empirical Evaluation of the GB2 Distribution

Since the GB2 distribution includes the lognormal and the log t as limiting cases, it will obviously do at least as well as these in estimating empirical security return distributions. The increased generality and flexibility of the GB2 distribution comes at the cost of additional parameters: four for the GB2 versus two for the lognormal and three for the log t. The purpose of this section is not to demonstrate the superiority of the empirical fit of the GB2 to these other two distributions, since that is mathematically apparent, but rather to give an indication of the amount of improvement in fit that is possible and to give an illustration of the typical parameter values that come out of the empirical use of the distribution.

Method of Data Generation: The Uses of Bootstrapping

The data used for the estimation procedure were generated from return data of the Center of Research in Security Prices tapes. Five hundred daily return observations were used, dating from December 30, 1981. Twenty-one randomly chosen stocks, listed with their ticker symbol in table 1, were used for the empirical tests. In order to test for distribu-

10. See Grandshteyn and Rhyzik (1965, p. 287, eq. [9]). The result also follows from Rainville (1960, p. 60).

TABLE 1 Stock Names and Ticker Symbols

Ticker Symbol	Stock Name	Ticker Symbol	Stock Name
AAC	Anacomp*	DFC	Dial Corp.
BEL	Bell Atlantic	DMG	DMG Inc.
BIW	Bath Iron Works†	DRV	Dravo Corp.
BNY	Bundy Corp.	EMP	Empire of California
BT	Bankers Trust	EXR	Elixir Industries
CAF	CNA Financial	FIS	Fischbach Corp.
CCF	Cook United	FTD	Fort Dearborn
CHR	Charter	GNS	Gilbert Systems Inc.‡
CLC	CLC of America	LTE	Electrosystems Inc.§
CPU	Compugraphic	NEG	New England Gas & Electric
CRF	Copeland Corp.		

* Went from OTC to NYSE.
† Now COG/Congoleum Corp. (as of January 1980).
‡ Now FLX/Flexi Van Corp.
§ Now ESY/*E*Systems Inc.

tional stationarity, for example, closure under multiplication, and to test for the convergence of the distributions to the lognormal as the return time period lengthened, we wish to consider returns over longer time periods of 25 and 250 days. Obviously, 500 observations are insufficient to allow meaningful tests over these longer time periods. While it would be possible to use a longer time period in the initial sample, doing so would lead to an increasing degree of heterogeneity in the sample. It is likely that a firm could take on a substantially different character over a 10- or 20-year period. This would lead to apparent nonstationarity in return distributions that could just as easily be regarded as an artifact of comparing stock returns from essentially different firms.

The problem, then, is obtaining observations for longer, nonoverlapping, time periods without imposing a large degree of nonstationarity by the use of data from disparate time periods. Several methods of generating approximate distributions for returns over longer time periods can be considered. These could include the use of bootstrapping or of a random number generator to generate larger samples from the estimated distribution of daily returns. A third approach could be based on deriving the sample moments for the longer time period from the corresponding sample moments for daily returns and then using method of moments to estimate a distribution for the longer time period. In this paper we use the bootstrap method. Similar related investigations using the random number generator approach provided results that were in general agreement with those obtained using the bootstrap. Additional research will consider the relative merits of the three techniques of generating return distributions for longer time periods. Bootstrapping assumes a given sample is representative of the overall population. It then reconstructs a larger sample by sampling with replacement from the given sample. If the initial sample is representative of the population, the bootstrapped sample will be a larger representative sample of the population. The technique of bootstrapping is growing in application when data limitations inhibit statistical tests.[11]

The resulting distribution approximations are asymptotically valid in a number of applications, such as when dealing with inferences on the mean (Babu and Singh 1983), with multiple linear regression models (Freedman 1981), and, as in the present application, with developing empirical approximations of the population distribution (Bickel and Freedman 1981; Singh 1981).

The procedure of bootstrapping in the current context can be outlined in the following steps. Suppose we want to generate 1,000 samples of 250-day returns. We take our initial sample of 500 observations and from it sample daily returns randomly with replacement 250,000 times.[12] The resulting bootstrapped sample can be regarded as one 250,000-element data set. We then multiply the first 250 of these resulting values together to get the first observation of 250-day returns, and we do the same with each of the following 250-day returns. The resulting 1,000-sample set of 250-day returns will have the same distributional properties as a sample of 250-day returns actually drawn from the population distribution from which the initial 500 samples were drawn.[13]

The properties of the bootstrap can be illustrated by using the bootstrap method to approximate a distribution that can also be represented empirically by a large sample set. A comparison of the bootstrap distribution to the actual distribution from which the bootstrap was drawn can then serve to indicate its accuracy. We have done this for 1-day and 5-day returns on the 500-day sample. We reconstructed the sample distribution by sampling with replacement from the data. In the case of the 5-day distribution, we then created 5-day returns by multiplying together each pair of five sampled data points. Treating the actual sample as the population from which the subset used for the bootstrap was drawn, we then compared the resulting bootstrapped distribution constructed from this subset of the data with the actual sample distribution. This comparison was done by using a Kolmogorov-Smirnov test on the two histograms.

11. For a general discussion of the bootstrap methods, see Efron (1979).

12. Each of the 500 daily returns would be taken as being equally likely.

13. It needs to be assumed that the daily returns are serially independent. The usual theoretical assumption that stock prices follow a martingale is sufficient for this, although a number of empirical studies have indicated some degree of serial correlation (e.g., Young 1971; Schwartz and Whitcomb 1977; and Perry 1982). If it is believed that such correlation is manifest only over smaller time intervals, one alternative to our approach would be to use longer time intervals—10-day intervals, e.g.—and sample these with replacement to obtain the desired 150-day sample.

TABLE 2 Tests of Bootstrap

Ticker Symbol	1-Day Maximum Difference	5-Day Maximum Difference
AAC	.071	.082
BEL	.022	.128
BIW	.034	.148
BNY	.022	.126
BT	.037	.108
CAF	.035	.088
CCF	.022	.214
CHR	.027	.088
CLC	.032	.120
CPU	.034	.070
CRF	.019	.098
DFC	.042	.168
DMG	.023	.206
DRV	.028	.068
EMP	.024	.840
EXR	.027	.088
FIS	.057	.050
FTD	.038	.144
GNS	.026	.076
LTE	.030	.104
NEG	.037	.098

The results of these tests for each of the 21 stocks are shown in table 2. The 5% rejection region for the 1-day distributions is .0860 and for the 5-day distributions is .1490. None of the 21 pairs of distributions for the 1-day returns can be rejected as being statistically different. Indeed, only two of the maximum differences for the 1-day difference are even half that required for rejection. Only three of the 21 distributions for the 5-day returns can be rejected as being statistically significantly different. The 5-day sample obviously has only one-fifth of the data available for the test, and the asymptotic properties of the bootstrap would not be as manifest.

Estimating Stock Returns with the GB2 Distribution

By applying the bootstrap technique, we estimated the distribution of 1-, 5-, 25-, and 250-day returns on 500 data points for each of the 21 stocks in the sample. The estimation criteria for fitting the empirical distributions with the GB2 and the lognormal distributions is the maximization of the log-likelihood function:

$$
\begin{aligned}
l_{GB2}(a, b, p, q) = {} & N \ln\{|a|[b^{ap}B(p, q)]\} + (ap - 1)\sum_{t=1}^{N} \ln(y_t) \\
& - (p + q)\sum_{t=1}^{N} \ln[1 + (y_t/b)^a].
\end{aligned} \tag{20}
$$

The maximum likelihood estimator will be asymptotically efficient if the true underlying population distribution is GB2. However, if the estimated distribution is a misspecification of the true underlying distribution, then other estimation methods may be more robust.[14] The results of estimating the GB2 and lognormal are given in the Appendix. A clarification needs to be made. The GB2, particularly for longer periods of time, can be thought of as an approximating distribution. We are not arguing closure. The GB2 with four parameters has sufficient flexibility to model the four data characteristics of greatest interest (mean, variance, skewness, and kurtosis) in a single parametric family. This flexibility is a major attraction of the GB2. The maximum likelihood estimation in this case could be viewed as a quasi maximum likelihood estimation technique and can still yield consistent estimates of data characteristics.

The log-likelihood function provides a natural test of comparative fit for the GB2 and the lognormal distributions. Twice the difference between the log-likelihood value of the estimate for the GB2 and the lognormal will be asymptotically distributed as a chi square with two degrees of freedom:

$$2[l(\Theta_{GB2}) - l(\Theta_{LN})] \sim \chi^2(2). \tag{21}$$

A summary of the chi-square values for the lognormal versus the GB2 for each of the estimates is presented in table 3. Since the test involves two degrees of freedom, a chi-square value greater than 10.6 indicates statistically superior performance for the GB2 distribution at the .995 confidence level. Nineteen of the 21 cases meet this criterion for 1-day returns based on the actual 500-sample data set, and 15 cases meet the criterion for the 1-day bootstrapped sample containing 500 observations. The number of cases that meet the criterion drops as the return period used increases. For a 5-day return, 10 of the 21 stocks show the GB2 to provide a better fit at the .995 confidence level when the actual prices are used, and 14 show a better fit when the bootstrapped sample is used.[15]

For the 25- and 250-day returns, the GB2 distribution does not provide as impressive an improvement in fit to the sample distribution as it

14. In particular, the method of moments may have a number of advantages over the log-likelihood estimator if issues of estimation of the first four moments, such as comparisons of kurtosis to test fat tails or estimates of variance for application to option models, are important. Alternative estimation methods that may be appropriate include minimizing the sum of squared errors, the absolute errors, or the chi-square criteria if the data are grouped.

15. In this case, the comparison of the actual and the bootstrapped data becomes more tenuous since the actual 5-day sample contains only 100 observations, compared to 500 for the bootstrapped sample. Because of the data limitations, no comparison of the actual and bootstrapped data was done for the 25- and 250-day returns. The results for 250-day returns should be interpreted with caution.

TABLE 3 Chi-Square Tests (H_0: GB2 = LN)

Stock Name	Real Data	1-Day Boot	5-Day Real	5-Day Boot	25-Day Boot	250-Day Boot
AAC	172.48	188.85	20.49	26.30	2.42	3.77
BEL	148.74	100.40	43.96	20.17	2.66	.66
BIW	112.36	75.91	.29	40.00	4.61	2.71
BNY	2.46	.28	.01	.85	0	1.27
BT	14.34	9.46	0	1.00	.42	1.75
CAF	78.62	41.66	.48	12.87	1.19	8.83
CCF	34.01	6.38	13.04	20.37	0	0
CHR	74.08	51.26	28.80	4.85	.02	0
CLC	87.12	17.06	11.38	36.70	16.73	.29
CPU	76.37	152.95	8.14	16.19	5.48	6.94
CRF	255.54	252.12	38.46	50.36	42.37	2.51
DFC	883.84	1,061.51	150.17	663.07	363.28	112.80
DMG	101.13	10.00	19.23	6.76	1.59	1.02
DRV	59.36	61.32	0	12.26	5.45	.26
EMP	4.70	21.20	5.52	19.86	8.06	3.91
EXR	84.42	39.02	16.12	5.18	8.68	1.50
FIS	102.40	100.24	.36	30.39	5.81	.12
FTD	79.40	6.36	.17	0	1.13	.63
GNS	239.19	311.02	21.28	52.63	34.24	1.26
LTE	26.30	8.36	.86	6.12	2.11	.58
NEG	57.82	78.98	.27	11.42	2.08	2.49
$\bar{X}$	127.80	126.8	17.57	47.71	23.20	7.00

does for the 1- and 5-day returns. Only four of the 21 distributions meet the .995 confidence level for the 25-day returns, and only six meet the .95 confidence level. For the 250-day returns, only one stock, DFC, meets the .995 confidence level, and only three stocks meet the .95 confidence level.

The obvious conclusion of this test of fit is that the GB2 does a significantly better job at fitting the sample distribution for returns over short periods, such as daily or weekly returns, and does not do significantly better as the period of returns used increases to monthly or yearly. Put in other terms, the short-term returns have a distribution that is significantly different from the lognormal and that still can be fit significantly better with the GB2 than with the lognormal; but over larger time periods the lognormal distribution becomes a better approximation in the sense that it does not do significantly worse at fitting the sample distribution than do any of the other distributions that make up the large set composed by the GB2. Such a conclusion for the behavior of the lognormal is consistent with the results of other studies, such as Blattberg and Gonedes (1974). Furthermore, it is an immediate result of the central limit theorem and the structure of the bootstrap procedure. Preliminary results using the random number generator approach to constructing annual return data confirm this property. This result has also been demonstrated in a security markets context by Arditti and Levy (1975).[16]

The central limit theorem assures that, as the size of the return period being considered becomes large, the distribution of resulting prices will converge to the lognormal distribution if the requisite regularity conditions are satisfied. The issues are whether these conditions are satisfied and how quickly convergence takes place. By employing the bootstrap methodology to generate a large sample of monthly and annual returns, our results indicate that such convergence is largely satisfied for annual return periods and to a large degree is already satisfied for monthly return periods as well.

V. Conclusion

The GB2 distribution is presented in this paper as a descriptive tool rather than as a definitive distribution. There is no law of nature that demands that security returns fit the GB2, any more than there is that the returns follow a log-*t* or a stable Paretian distribution. Like these distributions, the GB2 has attractive properties that should make it useful in empirical estimation of security returns and in some specialized theoretical work in which the specification of the distribution is of vital importance. The GB2 distribution is general and flexible, allowing tests between a number of competing return distributions that form special or limiting cases of the GB2; it can be expressed explicitly, facilitating the development of option pricing models and other models that depend on the specification and mathematical manipulation of distributions; and its parameter values have interpretations, such as the existence of higher moments, that can be useful in typifying the characteristics of return distributions.

For the sample of stocks we have considered, the GB2 provides a significantly better fit for returns, particularly over shorter time intervals, than the lognormal. The relative strength of the fit lends support to studies of returns arising from mixed distributions since the GB2 can be interpreted as the result of a broad class of mixed distributions. The interpretation of the GB2 distribution as a mixed distribution makes it useful as a tool for further investigation of the hypotheses that returns are generated by a mixed distribution.

16. The central limit theorem does not necessarily imply the normal distribution as the limiting distribution. Indeed, as is shown by Bartels (1977), the central limit theorem arguments can also lead to nonnormal stable limits in a number of cases familiar to the econometrician. The application of the central limit theorem in the case under consideration here is presented in a number of standard probability texts, such as Aitchison and Brown (1969, p. 13).

Appendix

TABLE A1 Parameter Estimates

File	Function	*F*-Value	Mean	Variance	Skewness	Kurtosis	Parameters	
							Lognormal: μ, σ	GB2: a, b, p, q
AAC.DAT	Sample		.100211D + 01	.403361D − 03	.173864D + 01	.152952D + 02	.191568D − 02	.268541D + 05
AAC.DAT	Lognormal	.125264D + 04	.100211D + 01	.390588D − 03	.591726D − 01	.300623D + 01	.388867D − 03	.999958D + 00
AAC.DAT	GB2	.133888D + 04	.100215D + 01	.324310D − 03	.471957D + 00	.627939D + 01		.273095D − 02
								.320725D − 02
BEL.DAT	Sample		.100303D + 01	.421934D − 03	.162429D + 01	.125576D + 02	.282100D − 02	.149721D + 04
BEL.DAT	Lognormal	.124117D + 04	.100303D + 01	.408943D − 03	.604920D − 01	.300651D + 01	.406394D − 03	.999201D + 00
BEL.DAT	GB2	.131554D + 04	.100300D + 01	.360310D − 03	.704770D + 00	.651970D + 01		.446458D − 01
								.588254D − 01
BIW.DAT	Sample		.100257D + 01	.647374D − 03	.258536D + 01	.290609D + 02	.225511D − 02	.724644D + 05
BIW.DAT	Lognormal	.114014D + 04	.100256D + 01	.612764D − 03	.740874D − 01	.300976D + 01	.609449D − 03	.999989D + 00
BIW.DAT	GB2	.119632D + 04	.100253D + 01	.572732D − 03	.462137D + 00	.631022D + 01		.770920D − 03
								.881871D − 03
BNY.DAT	Sample		.100025D + 01	.645446D − 03	−.731899D − 01	.471696D + 01	−.679843D − 04	.731804D + 01
BNY.DAT	Lognormal	.112646D + 04	.100026D + 01	.647282D − 03	.763222D − 01	.301036D + 01	.646742D − 03	.131953D + 01
BNY.DAT	GB2	.112769D + 04	.100026D + 01	.643598D − 03	−.657824D − 01	.301431D + 01		.249635D + 03
								.332315D + 02
BT.DAT	Sample		.100132D + 01	.315921D − 03	.145273D + 00	.371414D + 01	.116347D − 02	.161573D + 03
BT.DAT	Lognormal	.130651D + 04	.100132D + 01	.314836D − 03	.531662D − 01	.300503D + 01	.313956D − 03	.998988D + 00
BT.DAT	GB2	.131368D + 04	.100133D + 01	.327447D − 03	.334217D + 00	.507192D + 01		.509456D + 00
								.593092D + 00
CAF.DAT	Sample		.999758D + 00	.393130D − 03	−.170374D + 00	.625198D + 01	−.438691D − 03	.361014D + 06
CAF.DAT	Lognormal	.125006D + 04	.999759D + 00	.394643D − 03	.596191D − 01	.300632D + 01	.394755D − 03	.999999D + 00
CAF.DAT	GB2	.128937D + 04	.999756D + 00	.389584D − 03	.815665D − 01	.602321D + 01		.201561D − 03
								.195338D − 03
CCF.DAT	Sample		.100028D + 01	.180507D − 02	.141496D + 01	.102878D + 02	−.592255D − 03	.107639D + 01
CCF.DAT	Lognormal	.882607D + 03	.100027D + 01	.171948D − 02	.124438D + 00	.302754D + 01	.171709D − 02	.619406D + 01
CCF.DAT	GB2	.899613D + 03	.100024D + 01	.168323D − 02	.567730D + 00	.365230D + 01		.920312D + 03
								.905124D + 03

CHR.DAT	Sample		.998051D + 00	.135908D − 02	.883806D + 00	.719598D + 01	−.261992D − 02	.156604D + 03
CHR.DAT	Lognormal	.947871D + 03	.998046D + 00	.132364D − 02	.109408D + 00	.302129D + 01	.132794D − 02	.990669D + 00
CHR.DAT	GB2	.984975D + 03	.998029D + 00	.130570D − 02	.726867D + 00	.634281D + 01		.233862D + 00
								.303084D + 00
CLC.DAT	Sample		.100093D + 01	.957288D − 03	.104594D + 01	.860765D + 01	.463443D − 03	.287343D + 03
CLC.DAT	Lognormal	.103563D + 04	.100093D + 01	.931228D − 03	.914915D − 01	.301488D + 01	.929069D − 03	.994190D + 00
CLC.DAT	GB2	.107919D + 04	.100092D + 01	.917097D − 03	.792458D + 00	.659576D + 01		.147340D + 00
								.198179D + 00
CPU.DAT	Sample		.997978D + 00	.140089D − 02	−.323879D − 01	.689208D + 01	−.273037D − 02	.132985D + 04
CPU.DAT	Lognormal	.931457D + 03	.997981D + 00	.141366D − 02	.113077D + 00	.302274D + 01	.141837D − 02	.999051D + 00
CPU.DAT	GB2	.969642D + 03	.998572D + 00	.140456D − 02	.187023D + 00	.610303D + 01		.290010D − 01
								.277363D − 01
CRF.DAT	Sample		.100173D + 01	.262753D − 03	.380383D + 01	.464957D + 02	.160137D − 02	.177406D + 06
CRF.DAT	Lognormal	.136555D + 04	.100173D + 01	.248585D − 03	.472221D − 01	.300396D + 01	.247698D − 03	.999996D − 00
CRF.DAT	GB2	.149332D + 04	.100171D + 01	.174745D − 03	.465916D + 00	.624070D + 01		.559086D − 03
								.666192D − 03
DFC.DAT	Sample		.100270D + 01	.152247D − 02	.180240D + 02	.376733D + 03	.216525D − 02	.426033D + 06
DFC.DAT	Lognormal	.104123D + 04	.100262D + 01	.910582D − 03	.903181D − 01	.301451D + 01	.905416D − 03	.999998D + 00
DTC.DAT	GB2	.148315D + 04	.100230D + 01	.183179D − 03	.588803D + 00	.635668D + 01		.222347D − 03
								.281300D − 03
DMG.DAT	Sample		.999919D + 00	.115282D − 02	.570424D + 00	.470236D + 01	−.650458D − 03	.388218D + 05
DMG.DAT	Lognormal	.986466D + 03	.999916D + 00	.113396D − 02	.101070D + 00	.301817D + 01	.113351D − 02	.999979D + 00
DMG.DAT	GB2	.103703D + 04	.100019D + 01	.107024D − 02	.216530D + 00	.612583D + 01		.112231D − 02
								.110694D − 02
DRV.DAT	Sample		.999987D + 00	.333850D − 03	.293364D + 00	.628254D + 01	−.178897D − 03	.291002D + 03
DRV.DAT	Lognormal	.129321D + 04	.999987D + 00	.332056D − 03	.546740D − 01	.300531D + 01	.332009D − 03	.999182D + 00
DRV.DAT	GB2	.132289D + 04	.999984D + 00	.326262D − 03	.212158D + 00	.560911D + 01		.274903D + 00
								.288420D + 00
EMP.DAT	Sample		.100018D + 01	.383672D − 03	.600451D + 00	.656355D + 01	−.997770D − 05	.506673D + 01
EMP.DAT	Lognormal	.125999D + 04	.100018D + 01	.379274D − 03	.584217D − 01	.300607D + 01	.379066D − 03	.678822D + 00
EMP.DAT	GB2	.126234D + 04	.100030D + 01	.373652D − 03	.132006D − 02	.303622D + 01		.119647D + 03
								.849247D + 03

TABLE A1 (*Continued*)

File	Function	F-Value	Mean	Variance	Skewness	Kurtosis	Parameters: Lognormal: μ, σ	Parameters: GB2: a, b, p, q
EXR.DAT	Sample		.100196D + 01	.175711D − 02	.196041D + 01	.159832D + 02	.112651D − 02	.893148D + 02
EXR.DAT	Lognormal	.893531D + 03	.100195D + 01	.164578D − 02	.121534D + 00	.302627D + 01	.163804D − 02	.989620D + 00
EXR.DAT	GB2	.935739D + 03	.100190D + 01	.157207D − 02	.858465D + 00	.619531D + 01		.377438D + 00
								.558261D + 00
FIS.DAT	Sample		.100105D + 01	.242441D − 03	.763417D + 00	.578023D + 01	.934568D − 03	.155063D + 06
FIS.DAT	Lognormal	.137497D + 04	.100105D + 01	.239393D − 03	.463718D − 01	.300382D + 01	.238861D − 03	.999997D + 00
FIS.DAT	GB2	.142617D + 04	.100104D + 01	.226655D − 03	.297950D + 00	.612707D + 01		.582515D − 03
								.635823D − 03
FTD.DAT	Sample		.100022D + 01	.240813D − 03	.219372D − 01	.383522D + 01	.971608D − 04	.269379D + 06
FTD.DAT	Lognormal	.137385D + 04	.100022D + 01	.240468D − 03	.465147D − 01	.300385D + 01	.240334D − 03	.999999D + 00
FTD.DAT	GB2	.141355D + 04	.100014D + 01	.237282D − 03	.119124D + 00	.603458D + 01		.340678D − 03
								.341248D − 03
GNS.DAT	Sample		.100137D + 01	.121946D − 02	.411824D + 01	.548636D + 02	.804767D − 03	.100350D + 05
GNS.DAT	Lognormal	.994597D + 03	.100135D + 01	.109761D − 02	.992924D − 01	.301753D + 01	.109404D − 02	.993903D + 00
GNS.DAT	GB2	.111419D + 04	.100124D + 01	.829439D − 03	.929234D + 00	.700125D + 01		.427929D − 02
								.609986D − 02
LTE.DAT	Sample		.100084D + 01	.525004D − 03	−.186605D + 00	.469219D + 01	.579956D − 03	.112989D + 03
LTE.DAT	Lognormal	.117757D + 04	.100084D + 01	.527490D − 03	.688554D − 01	.300843D + 01	.526462D − 03	.100054D + 01
LTE.DAT	GB2	.119072D + 04	.100084D + 01	.527342D − 03	.131911D + 00	.477957D + 01		.629263D + 00
								.630690D + 00
NEG.DAT	Sample		100063D + 01	.274691D − 03	−.113826D + 00	.424056D + 01	.497474D − 03	.261262D + 05
NEG.DAT	Lognormal	.134034D + 04	.100063D + 01	.274975D − 03	.497201D − 01	.300440D + 01	.274588D − 03	.100000D + 01
NEG.DAT	GB2	.136925D + 04	.100064D + 01	.283771D − 03	.214229D + 00	.608325D + 01		.315413D − 02
								.328763D − 02

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