Equal sacrifice taxation

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Abstract

We axiomatically characterize the family of equal sacrifice rules for the problem of fair taxation: every agent with positive post-tax income sacrifices the same amount of utility relative to his/her respective pre-tax income. In contrast to the result in Young (1988), our family of rules allows for asymmetric and "constrained" versions of equal sacrifice. When we add the requirement that an agent's tax burden must not decrease when their income increases, then this is equivalent to assuming that every agent's utility function is concave. When we add the requirement that a tax rule be independent of scale, then this is equivalent to assuming that every agent has the same constant measure of relative risk aversion. In addition, as a special case of our family of rules, we derive a tighter result than Young (1988) by showing one of his axioms is unnecessary.

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Equality of taxation, therefore, as a maxim of politics, means equality of sacrifice. It means apportioning the contribution of each person towards the expenses of government, so that he shall feel neither more nor less inconvenience from his share of the payment than every other person experiences from his. This standard, like other standards of perfection, cannot be completely realized; but the first object in every practical discussion should be to know what perfection is.

–John Stuart Mill, *Principles of Political Economy*

1. Introduction

Consider the problem of fair taxation: Given a fixed amount of tax revenue that needs to be raised, and given the amount of income that each citizen has, how much should each citizen be taxed? This has long been a problem of interest to philosophers, economists, and politicians. Indeed, discussions of fair taxation in the public sphere often follow any proposal to modify the tax system.

One method of fair taxation proposed by John Stuart Mill, as quoted in the epigraph, is to impose an equal amount of subjective sacrifice on each individual. In this paper, we adopt this principle of equal sacrifice in taxation and axiomatically characterize all such taxation methods. The goal is to interpret the equal sacrifice principle as broadly as is reasonable, and thus come to a better understanding of the underlying properties of Mill's "standard of perfection."

The first axiomatic study of the equal sacrifice principle applied to fair taxation is Young (1988). In that paper, Young considers a family of taxation methods (called rules) that assign taxes as follows. A member of this family is defined by a utility function U over income which is continuous, strictly increasing, and unbounded from below. The rule allocates taxes so that each individual's utility loss according to U is the same. That is, for individuals i and j with pre-tax incomes c_i and c_j and post-tax incomes x_i and x_j, we have

$$U(c_i) - U(x_i) = U(c_j) - U(x_j).$$

Note two properties of this family of rules and how they relate to restrictions on the utility functions. First, such a rule is symmetric in the sense that the same U is applied to all individuals. Second, the rule is unconstrained in the sense that it is always able to equalize sacrifice across all individuals since U is unbounded from below;[1] there is never an instance in which the rule must impose less sacrifice on an individual because no more taxes can be extracted from her.

In this paper, we consider a more general family of equal sacrifice rules that improves on Young's by dropping these two unnecessary restrictions on the utility functions used to calculate sacrifice, thus allowing for both asymmetric and constrained rules. A member of this family is defined by a collection of utility functions $\{U_i\}$, one for each individual, where each U_i is continuous and strictly increasing (but not necessarily unbounded from below). The rule allocates taxes so that each individual receiving strictly positive post-tax income will have the same utility loss, while each individual receiving zero post-tax income will receive less utility loss. That is, for individuals i, j, and k with pre-tax incomes c_i, c_j, and c_k, if the rule allocates post-tax incomes $x_i, x_j > 0$ and $x_k = 0$, then we must have

$$U_i(c_i) - U_i(x_i) = U_j(c_j) - U_j(x_j) > U_k(c_k) - U_k(x'_k) \ \ \forall x'_k \in (0, c_k].$$

This allows for asymmetric rules since the utility functions for the individuals are potentially different. In addition, the rule may be constrained since one individual may experience less sacrifice than the others because the rule cannot assign a tax more than an individual's income.

Allowing for asymmetric equal sacrifice rules is a natural extension of Young's family of rules. However, allowing for asymmetry of utility may also be desirable for normative reasons. That is, for reasons of fairness, the taxing authority may want to treat two individuals differently simply because they have different needs and situations. Indeed, in the United States, one's tax burden is determined by more than just pre-tax income, such as the number of dependents the individual has. Given this observation, one approach would be to extend the model to include all relevant information the taxing authority uses to determine the assignment of taxes. However, to keep the model broadly applicable to other contexts, as well as to make it easily comparable to the existing literature, we keep the standard framework wherein only identities and pre-tax income are used to determine the assignment of taxes.

Allowing for constrained versions of equal sacrifice rules by dropping the unboundedness from below condition is desirable because it makes the equal sacrifice family more general. Indeed, the simplest form of equal sacrifice is to impose the same tax on each individual. However, to be

[1] For any utility loss λ, one can always find a post-tax income level x_i such that $U(c_i) - U(x_i) = \lambda$.

valid, this rule must be constrained since an agent cannot be taxed more than her income. In the literature, this rule is referred to as the constrained equal loss rule, though in the context of taxation it is commonly called the head tax. Young's family of rules excludes this important equal sacrifice rule while ours permits it.

Our main theorem, Theorem 1, axiomatically characterizes this family of rules. Our two most important axioms are prominent in the literature. The first, Consistency, says that how a rule assigns taxes does not change when the group to be taxed shrinks coupled with an appropriate shrinking of the tax burden. The second, Composition Down, says that if the total tax burden increases, then it is sufficient to use current income (i.e. the post-tax income under the previous, smaller tax burden) to determine the new assignment of taxes.

In addition, we impose a novel axiom which is weaker than Strict Claims Monotonicity, a well-known axiom in the literature.[2] Strict Claims Monotonicity states that if one individual's pre-tax income increases, then her post-tax income should increase. We impose this requirement as well, but only in instances in which every individual has positive post-tax income. We call our axiom Positive Awards Strict Claims Monotonicity.

Given the prominence of concave utility functions in economic theory, a natural question is what implications concavity would have on our division rule. Theorem 2 shows that adding an axiom called Bounded Gain from Linked Claim-Endowment Increase to the set of axioms from Theorem 1 is equivalent to adding the requirement that the utility functions $\{U_i\}$ all be concave. In the context of taxation, Bounded Gain from Linked Claim-Endowment Increase is simply the requirement that when one individual's pre-tax income increases while the tax burden stays constant, then that individual's tax burden must not decrease.

Theorem 3 shows the implications of adding the axiom Homogeneity to Theorem 1. Homogeneity states that a how a rule allocates taxes should be independent of scale. Interestingly, when this axiom is added, every agent must exhibit the same constant measure of relative risk aversion.

One common axiom obviously missing from Theorem 1 is Equal Treatment of Equals, which states that two individuals with equal income will be taxed equally. Theorem 5 shows that when we add Equal Treatment of Equals to our set of axioms, the result is a generalization of Young's

[2]The term "claims" is borrowed from the conflicting claims literature, of which the current work is a part. We discuss this further, and our nomenclature in general, in section 2.

family of symmetric equal sacrifice rules that allows for constrained rules. Thus one contribution of this paper is simply a better understanding of the logical implications of Equal Treatment of Equals. That is, because Equal Treatment of Equals invariably plays a central role in the proof of any theorem that employs it, an important question is what happens without it. Theorem 1 and Theorem 5 together demonstrate that relaxing Equal Treatment of Equals (in the presence of our other axioms) does nothing more than allow for different utility functions for the agents.[3]

Corollary 1 shows that if, in addition to adding Equal Treatment of Equals, we strengthen Positive Awards Strict Claims Monotonicity to Strict Claims Monotonicity, the result is exactly Young's family of symmetric and unconstrained equal sacrifice rules. This alternative characterization is tighter than Young's result as we do not assume Strict Endowment Monotonicity as he did.

Besides Young's paper, there are two other papers in the literature closely related to the present work. Chambers and Moreno-Ternero (2017) consider a generalized family of symmetric equal sacrifice rules that allows for constrained versions of Young's family. Theorem 5 is a special case of their main result. Naumova (2002) considers asymmetric equal sacrifice rules, but only ones that are unconstrained. However, Naumova's setting is significantly different from the canonical one we consider, and so it is difficult to compare results. Our Theorem 4 is an analogue to her result.

More broadly, the current work (like Naumova's) adds to the growing literature studying rules that do not impose Equal Treatment of Equals. The seminal paper here is Moulin (2000), but Chambers (2006), Flores-Szwagrzak (2015), Harless (2017), Hokari and Thomson (2003), Kıbrıs (2012, 2013), Moulin (2000), and Stovall (2014a,b) all consider rules that are (possibly) asymmetric. Of these papers, only Stovall (2014a) is easily relatable to the family we characterize, though the overlap with Moulin's family is substantial. We discuss closely related papers in more detail in section 6. For readers wishing to preview the relation between these other papers and the current work, Table 1 and Figure 1 provide summaries.

Formally, the problem of fair taxation is identical to the problem of fair allocation under conflicting claims: Given a fixed endowment that must be divided among a group, each individual of the group having some (objective) claim on the endowment, and given that the amount to be divided is not sufficient to satisfy all claims, how should the endowment be divided? Other examples

[3]See Stovall (2014a) for an example in which relaxing Equal Treatment of Equals does not lead to a straightforward result.

of conflicting claims problems are bankruptcy and cost sharing. Modern study of claims problems began with O'Neill (1982). See Thomson (2019) for a comprehensive review of this literature.[4]

We discuss the relation between fair taxation problems and conflicting claims problems further in section 2. We define and discuss the family of equal sacrifice rules in section 3 and our main axioms in section 4. We present our main results in section 5. We conclude in section 6 by discussing related literature. All proofs are relegated to the appendix.

2. Taxation problems and duality

We use the following notation. Let $\mathcal{N}$ denote the set of finite and non-singleton subsets of the natural numbers, $\mathbb{N}$. Let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the non-negative real numbers and the positive real numbers respectively. Let $\mathbf{0}$ denote a vector of zeros. For $x, y \in \mathbb{R}^N$, we use the vector inequalities $x \geqq y$ if $x_i \geq y_i$ for all $i \in N$, $x \geq y$ if $x \geqq y$ and $x \neq y$, and $x > y$ if $x_i > y_i$ for every $i \in N$. For $x \in \mathbb{R}^N$ and $N' \subset N$, let $x_{N'}$ denote the projection of x onto the subspace $\mathbb{R}^{N'}$. For $i \in N$, let x_{-i} denote $x_{N \setminus \{i\}}$.

A *problem* is a tuple (N, c, E) where $N \in \mathcal{N}$, $c \in \mathbb{R}^N_{++}$, and $E \in [0, \sum_N c_i]$. An *award* for the problem (N, c, E) is an N-vector x satisfying $\mathbf{0} \leqq x \leqq c$ and $\sum_N x_i = E$. A *rule* is a function S that maps problems to awards.

In the context of taxation, we think of c_i as being agent i's pre-tax income and E as representing the total amount of post-tax income. Thus $T = \sum_N c_i - E$ is the total amount of tax to be collected. The requirement $E \leq \sum_N c_i$ then says that some tax will be raised (i.e. $T \geq 0$), while the requirement $E \geq 0$ says that total tax cannot exceed national income (i.e. $T \leq \sum_N c_i$). An award x_i for agent i is the amount of post-tax income that i gets, making $t_i = c_i - x_i$ her tax burden. Thus the requirement $x_i \geq 0$ says that an agent cannot be taxed more than her income (i.e. $t_i \leq c_i$), while the requirement $x_i \leq c_i$ says that an agent's income cannot be subsidized by tax revenue (i.e. $t_i \geq 0$). Finally the requirement that $\sum_N x_i = E$ combines the feasibility requirement ($\sum_N x_i \leq E$) and the efficiency requirement ($\sum_N x_i \geq E$).

As mentioned in the introduction, a taxation problem is formally equivalent to the problem of fair allocation under conflicting claims, the most prominent example of such a problem being bankruptcy. In this context, c_i is agent i's claim on the endowment, while E is the total amount of

[4] See Thomson (2003, 2015) for shorter surveys.

the endowment to be divided. The requirement that $E \leq \sum_N c_i$ says that there is not enough of the endowment to satisfy everyone's claim on it. We usually think of E as representing a resource that is desirable for all agents, though this is not necessary. Indeed, an alternative way of thinking about a problem is not how to divide the endowment, but rather how to divide the loss among the agents. That is, $\sum_N c_i - E$ represents the total loss, or shortage, that must be divided. This alternative way of thinking about a problem brings us to the following definitions. The *dual of a problem* (N, c, E) is the problem $(N, c, \sum_N c_i - E)$. The *dual of a rule* S is the rule S^d satisfying $S^d(N, c, E) = c - S(N, c, \sum_N c_i - E)$ for every problem (N, c, E). The *dual of an axiom* A is the axiom A^d such that S satisfies A if and only if S^d satisfies A^d. An axiom A is *self-dual* if $A^d = A$.

Since our ultimate goal is to study fair taxation, it may seem like a roundabout approach to study rules that allocate post-tax income rather than rules that allocate taxes directly. However, to make our results more readily comparable to the literature on conflicting claims, we adopt the perspective that the endowment to be divided, E, is desirable for the agents.[5] Thus E represents the total amount of post-tax income, while a rule S allocates this post-tax income. Given the definitions above, it is a straightforward step to go from studying income allocation rules to studying tax allocation rules. That is, if S is a post-tax income allocation rule that satisfies axiom A, then S^d is a tax allocation rule that satisfies axiom A^d.

In addition, because the conflicting claims framing is more prominent in the literature, we adopt the terminology of conflicting claims problems when naming axioms. For example, we use 'endowment', 'claims', and 'award' instead of 'tax burden', 'pre-tax income', and 'post-tax income', respectively.[6]

3. Equal sacrifice rules

To define our family of rules formally, we introduce some notation. Let $\mathcal{U}$ denote the family of functions $U : \mathbb{N} \times \mathbb{R}_{++} \to \mathbb{R}$ such that, for any $i \in \mathbb{N}$, $U(i, \cdot)$ is continuous and strictly increasing.

[5] We note that while Young (1988) takes the opposite perspective (i.e. the endowment to be divided is the total tax revenue, and is thus undesirable), Naumova (2002) and Chambers and Moreno-Ternero (2017) share our perspective in framing a taxation problem.

[6] For all but our novel axioms, we will generally borrow axiom names from Thomson (2019). There is one exception to this. See footnote 12.

For ease of notation, from now on we write $U(i, \cdot)$ as U_i. Note that we make no assumption about the value of $\lim_{x_i \to 0} U_i(x_i)$. However, we say U_i is *unbounded from below* if $\lim_{x_i \to 0} U_i(x_i) = -\infty$.

For $U \in \mathcal{U}$ and $i \in \mathbb{N}$, set $\underline{u}_i \equiv \lim_{x_i \to 0} U_i(x_i)$ and $\bar{u}_i \equiv \lim_{x_i \to \infty} U_i(x_i)$. Since U_i is continuous and strictly increasing, it is invertible over $(\underline{u}_i, \bar{u}_i)$. Let $U_i^{-1} : (\underline{u}_i, \bar{u}_i) \to \mathbb{R}_{++}$ denote the inverse function of U_i. Let $\overline{U_i^{-1}}$ denote the left-hand extension of U_i^{-1}:

$$\overline{U_i^{-1}}(u) \equiv \begin{cases} 0 & \text{if } u \leq \underline{u}_i, \\ U_i^{-1}(u) & \text{if } \underline{u}_i < u < \bar{u}_i. \end{cases}$$

Note that $\overline{U_i^{-1}}$ is continuous and weakly increasing. In particular, $\overline{U_i^{-1}}$ is constant on $(-\infty, \bar{u}_i]$ and strictly increasing on $[\underline{u}_i, \bar{u}_i)$.

For $U \in \mathcal{U}$, we define the *equal sacrifice rule relative to* U, denoted ES^U, as follows. For any problem (N, c, E),

$$ES^U(N, c, E) \equiv \left(\overline{U_i^{-1}} \left(U_i(c_i) - \lambda \right) \right)_{i \in N},$$

where $\lambda \geq 0$ is chosen so that $\sum_{i \in N} \overline{U_i^{-1}} \left(U_i(c_i) - \lambda \right) = E$.[7] We say a rule S is an *equal sacrifice rule* if there exists $U \in \mathcal{U}$ such that $S = ES^U$. We say that U is an *equal sacrifice representation of* ES^U. We use $\mathcal{ES}$ to denote the family of equal sacrifice rules. I.e.

$$\mathcal{ES} \equiv \left\{ ES^U : U \in \mathcal{U} \right\}.$$

We introduce a few special cases of equal sacrifice rules. We say that an equal sacrifice rule ES^U is *constrained* if there exists (N, c, E) such that $E > 0$ and $ES_i^U(N, c, E) = 0$ for some $i \in N$. Note that for this to be true, we must have $\lim_{x_i \to 0} U_i(x_i) \neq -\infty$. Therefore we define the family of *unconstrained equal sacrifice rules* to be

$$\widehat{\mathcal{ES}} \equiv \left\{ ES^U : U \in \mathcal{U} \text{ and } \lim_{x \to 0} U_i(x) = -\infty \text{ for all } i \in \mathbb{N} \right\}.$$

The family of *symmetric equal sacrifice rules* is

$$\mathcal{ES}^* \equiv \left\{ ES^U : U \in \mathcal{U} \text{ and } U_i = U_j \text{ for all } i, j \in \mathbb{N} \right\}.$$

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[7]Note that ES^U is well-defined and a rule: For any $i \in \mathbb{N}$, $c_i > 0$, and $\lambda \geq 0$, we must have $0 \leq \overline{U_i^{-1}} \left(U_i(c_i) - \lambda \right) \leq c_i$. Also, note that for any $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, $F(\lambda) \equiv \sum_{i \in N} \overline{U_i^{-1}} \left(U_i(c_i) - \lambda \right)$ is continuous and strictly decreasing, $F(0) = \sum_{i \in N} c_i$, and $\lim_{\lambda \to \infty} F(\lambda) = 0$. Thus for $E > 0$, there exists a unique $\lambda^* \geq 0$ such that $F(\lambda^*) = E$. For $E = 0$, then it is possible that there exists λ' and λ'' such that $F(\lambda') = F(\lambda'') = 0$. However, both λ' and λ'' would assign the same award, namely 0 for everyone.

Note that the family of symmetric unconstrained equal sacrifice rules, $\widehat{\mathcal{ES}}^* \equiv \widehat{\mathcal{ES}} \cap \mathcal{ES}^*$, is the family characterized by Young (1988).

The family of equal sacrifice rules contains some prominent rules. The *proportional rule*, P, allocates post-tax income proportionally to pre-tax income:

$$P(N, c, E) = \frac{E}{\sum_N c_i} c.$$

This is commonly referred to as the *flat tax*, where $1 - \frac{E}{\sum_N c_i}$ is the tax rate. The proportional rule is an equal sacrifice rule where $U_i(x_0) = U_j(x_0) = \ln x_0$ for every $i, j \in \mathbb{N}$. Thus $P \in \widehat{\mathcal{ES}}^*$.

The *constrained equal loss rule*, CEL, imposes the same loss (i.e. tax) on every individual as long as that tax is not more than their respective income. So for $i \in N$,

$$CEL_i(N, c, E) = \max\{0, c_i - \lambda\},$$

where λ (the common tax imposed on everyone) is chosen so that $\sum_N CEL_i(N, c, E) = E$. This is commonly referred to as the *head tax*. The constrained equal loss rule is an equal sacrifice rule where $U_i(x_0) = U_j(x_0) = x_0$ for every $i, j \in \mathbb{N}$. Thus $CEL \in \mathcal{ES}^*$.

The *weighted constrained equal loss rule with weights* $w \in \mathbb{R}^{\mathbb{N}}_{++}$, $WCEL^w$, is an asymmetric version of the constrained equal loss rule.[8] This rule assigns a weight w_i to each agent and then equalizes the weighted losses as long as that does not imply a loss larger than an agent's income. So for $i \in N$,

$$WCEL_i^w(N, c, E) = \max\{0, c_i - w_i\lambda\},$$

where λ is chosen so that $\sum_N WCEL_i^w(N, c, E) = E$. The weighted constrained equal loss rule is an equal sacrifice rule where $U_i(x_i) = \frac{x_i}{w_i}$ for every $i \in \mathbb{N}$. Thus generally, $WCEL^w \in \mathcal{ES}$ for $w \in \mathbb{R}^{\mathbb{N}}_{++}$.

An important question regarding the equal sacrifice rules is to what extent an equal sacrifice representation is unique. This is answered by our first result.

Proposition 1. *Suppose $U \in \mathcal{U}$. Then $V \in \mathcal{U}$ is an equal sacrifice representation of ES^U if and only if there exist $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}^{\mathbb{N}}$ such that $V_i = \alpha U_i + \beta_i$ for every i.*

[8]See Moulin (2000) and Flores-Szwagrzak (2015).

Thus an equal sacrifice rule is invariant to common changes of units and independent changes of origins. This means that an equal sacrifice representation does not have to take a stand on a relative zero for the agents, but it does have to take a stand on a relative unit.

4. Axioms

4.1. Main axioms

Our first three axioms are standard in the literature.

Continuity. For each problem (N, c, E) and each sequence of problems $\{(N, c^m, E^m)\}$, if $(N, c^m, E^m) \to (N, c, E)$, then $S(N, c^m, E^m) \to S(N, c, E)$.

Continuity simply requires that the rule be jointly continuous in total post-tax income and the vector of pre-tax incomes.

Consistency. For each problem (N, c, E) and each $N' \subset N$, if $x \equiv S(N, c, E)$, then $x_{N'} = S(N', c_{N'}, \sum_{N'} x_j)$.

Consistency imposes a restriction on the rule when the group shrinks. It says that how a rule assigns post-tax income among a subpopulation should not change when considered as a separate problem, fixing the total amount of post-tax income for that subpopulation.

Composition Down. For each problem (N, c, E) where $S(N, c, E) > \mathbf{0}$, and each $E' \in [0, E)$, we have $S(N, c, E') = S(N, S(N, c, E), E')$.

Imagine a scenario in which total post-tax income was determined to be E. Citizens subsequently pay their respective assigned tax, leaving the post-tax allocation $S(N, c, E)$. Then it is discovered that the requisite tax revenue is larger than initially determined, decreasing total post-tax income to E'. Composition Down says that the new post-tax income allocation can be determined either by using everyone's original income (i.e. c) or their previous post-tax income (i.e. $S(N, c, E)$); both methods will yield the same result.

Our final main axiom is, to our knowledge, new to the literature.

Positive Awards Strict Claims Monotonicity (PASM-Claims). For each problem (N, c, E) where $S(N, c, E) > \mathbf{0}$, each $i \in N$, and each $c'_i > c_i$, we have $S_i(N, (c'_i, c_{-i}), E) > S_i(N, c, E)$.

PASM-Claims imposes the requirement that an agent's post-tax income must increase if her pre-tax income increases, but only in scenarios in which everyone was receiving positive post-tax income.

4.2. Discussion of axioms

PASM-Claims is closely related to two prominent axioms.

Claims Monotonicity. For each problem (N, c, E), each $i \in N$, and each $c'_i > c_i$, we have $S_i(N, (c'_i, c_{-i}), E) \geq S_i(N, c, E)$.

Strict Claims Monotonicity. For each problem (N, c, E) where $E > 0$, each $i \in N$, and each $c'_i > c_i$, we have $S_i(N, (c'_i, c_{-i}), E) > S_i(N, c, E)$.

Claims Monotonicity says that if an agent's pre-tax income increases, then that agent's post-tax income should not decrease. Strict Claims Monotonicity requires post-tax income to strictly increase if pre-tax income increases.

It is easy to see that all equal sacrifice rules satisfy Claims Monotonicity since each utility function is strictly increasing. However, Strict Claims Monotonicity precludes equal sacrifice rules that are constrained. This is because a constrained rule allows for one agent to get zero post-tax income while other agents get positive post-tax income. But rules satisfying Strict Claims Monotonicity must award positive post-tax income to all agents when $E > 0$.

PASM-Claims is weaker than Strict Claims Monotonicity and allows for constrained rules. The key to understanding PASM-Claims is the condition $S(N, c, E) > \mathbf{0}$. Note that when $S(N, c, E) \not> \mathbf{0}$, then the strict monotonicity condition does not have to hold. Thus scenarios where $S_i(N, (c'_i, c_{-i}), E) = S_i(N, c, E) = 0$ are allowed. This is desirable because if a problem is such that agent i gets no post-tax income, then there may be good reason to continue giving her no post-tax income if her pre-tax income slightly increases. Another reason why the condition $S(N, c, E) > \mathbf{0}$ is important is because it requires that there be other agents who are not i from which to transfer post-tax income to agent i. It would be impossible to have $S_i(N, (c'_i, c_{-i}), E) > S_i(N, c, E)$ if $S_i(N, c, E) = E$.

Instead of looking at increases in pre-tax income, another set of prominent axioms deal with monotonicity of awards relative to the total post-tax income available.

Endowment Monotonicity. For each problem (N, c, E) and each $E' \in (E, \sum_N c_j]$, we have $S(N, c, E') \geq S(N, c, E)$.

Strict Endowment Monotonicity. For each problem (N, c, E) and each $E' \in (E, \sum_N c_j]$, we have $S(N, c, E') > S(N, c, E)$.

Similar to Claims Monotonicity, monotonicity of the utility functions imply all equal sacrifice rules satisfy Endowment Monotonicity, while Strict Endowment Monotonicity preclude constrained equal sacrifice rules. Now consider a modification of these axioms in the same spirit as PASM-Claims.[9]

Positive Awards Strict Endowment Monotonicity (PASM-Endowment). For each problem (N, c, E) where $S(N, c, E) > \mathbf{0}$, and each $E' \in (E, \sum_N c_j]$, we have $S(N, c, E') > S(N, c, E)$.

PASM-Endowment says that as long as everyone has positive post-tax income, then increases in total post-tax income (i.e. decreases in the tax burden) will strictly benefit all agents. Interestingly, in conjunction with the other main axioms, PASM-Claims implies PASM-Endowment, and so we do not include it in our main result.[10]

Together, PASM-Claims and PASM-Endowment imply an interesting property that has a nice interpretation for the fair taxation setting. Namely, they imply that for each problem (N, c, E) where $S(N, c, E) > \mathbf{0}$, each $i \in N$, and each $h > 0$, we have $S_i(N, (c_i + h, c_{-i}), E + h) > S_i(N, c, E)$. The interpretation of this property is that each agent's marginal tax rate must be strictly less than 100% when the tax burden is relatively low. To see this, note that the total tax revenue is the same under the problems $(N, (c_i + h, c_{-i}), E + h)$ and (N, c, E) since

$$\left(\sum_N c_i + h\right) - (E + h) = \sum_N c_i - E.$$

Thus the difference between $(N, (c_i + h, c_{-i}), E + h)$ and (N, c, E) is that i's pre-tax income has increased while the total tax burden remains constant. This condition then implies that i must strictly benefit in this instance when everybody has positive post-tax income.

5. Results

Our main result characterizes the family $\mathcal{ES}$.

[9] Thomson (2019, section 4.1) introduces an axiom called "Null-Compensation–Conditional Strict Endowment Monotonicity" that is similar in spirit to PASM-Endowment.

[10] See Lemma 2.

Theorem 1. *The rule S satisfies Continuity, Consistency, Composition Down, and PASM-Claims if and only if S is an equal sacrifice rule.*

The following examples demonstrate the extent to which the listed axioms are independent. For each axiom below, we give a rule which violates that axiom but satisfies the others in Theorem 1.

- **Consistency.** A rule that divides according to the proportional rule for all two-person groups and according to the constrained equal loss rule for all groups larger than two.

- **Composition Down.** The symmetric parametric rule, S, with the parametric function[11]

$$f(c_0, \lambda) = \begin{cases} \frac{c_0}{1-\lambda c_0} & \text{if } \lambda < -\frac{1}{c_0}, \\ \frac{c_0}{2} & \text{if } -\frac{1}{c_0} \leq \lambda \leq \frac{1}{c_0}, \\ c_0 - \frac{c_0}{1+\lambda c_0} & \text{if } \lambda > \frac{1}{c_0}. \end{cases}$$

  Because S is a symmetric parametric rule, it satisfies Continuity and Consistency. Also, because f is strictly increasing in c_0, S must satisfy PASM-Claims. However, S does not satisfy Composition Down. To see this, note that $S(\{1,2\},(6,2),4) = (3,1)$, yet

$$\left(\frac{3}{2}, \frac{1}{2}\right) = S(\{1,2\},(3,1),2) \neq S(\{1,2\},(6,2),2) = \left(\frac{6}{2+\sqrt{10}}, \frac{6}{4+\sqrt{10}}\right).$$

- **PASM-Claims.** The *constrained equal awards rule*. This tax assigns the same post-tax income to all agents, with the proviso that no agent's post-tax income is more than their respective pre-tax income. In the taxation context, this rule is referred to as the *leveling tax*.

It is an open question whether Continuity is independent of the other axioms. However, we note that it is possible to show that Composition Down implies continuity in the total post-tax income E. Thus the only question is whether continuity in the pre-tax income vector c is implied by the other axioms.

The family of equal sacrifice rules is designed to capture in the most general way possible the equal sacrifice principle proposed by John Stuart Mill. A natural line of inquiry then is how strengthening the axioms in Theorem 1 introduces restrictions on the admissible utility functions used to calculate sacrifice. We explore this in a series of results.

[11] The family of symmetric parametric rules is characterized by Young (1987). See section 6 for a discussion of this family.

We first examine what is needed to guarantee that the equal sacrifice representation of a rule is concave.[12]

Bounded Gain from Linked Claim-Endowment Increase. For each problem (N, c, E), each $i \in N$, and each $h > 0$, we have $h \geq S_i(N, (c_i + h, c_{-i}), E + h) - S_i(N, c, E)$.

It is easy to show that Bounded Gain from Linked Claim-Endowment Increase is the dual to Claims Monotonicity.[13] Thus in the context of taxation, Bounded Gain from Linked Claim-Endowment Increase says that if an agent's pre-tax income increases while the total tax burden stays constant, then that agent's personal tax burden must not decrease. This is enough to guarantee concave utility functions.[14]

Theorem 2. *The rule S satisfies Continuity, Consistency, Composition Down, PASM-Claims, and Bounded Gain from Linked Claim-Endowment Increase if and only if S is an equal sacrifice rule in which every agent's utility function is concave.*

Our next result explores the implications of adding a common scale invariance property to Theorem 1.

Homogeneity. For each problem (N, c, E) and each $\lambda > 0$, we have $S(N, \lambda c, \lambda E) = \lambda S(N, c, E)$.

Though often defended as saying that the rule should be independent of the unit of account, Homogeneity is in fact much stronger. It says that the rule should be independent of scale; that problems involving pennies should be decided in the same manner as problems involving millions of dollars.

Homogeneity imposes a significant amount of structure to equal sacrifice rules. Recall that *the measure of relative risk aversion for U_0 at x_0* is $-\frac{U_0''(x_0)}{U_0'(x_0)}$. If this is constant for all x_0, then we say the agent has a *constant measure of relative risk aversion.* Any increasing utility function U_0 with

[12] This axiom is called "Linked Claim-Endowment Monotonicity" by Thomson (2019). We have chosen a different name as "Linked Claim-Endowment Monotonicity" would describe the following property: For each problem (N, c, E), each $i \in N$, and each $h > 0$, we have $S_i(N, (c_i + h, c_{-i}), E + h) \geq S_i(N, c, E)$. This property is referenced, but not named, in Thomson (2019, section 7.2).

[13] See Thomson and Yeh (2008) for further discussion of this result, as well as the duality operator in general.

[14] In his book, Young (1994, p.106) alludes to (but does not prove) a similar result for his family of symmetric unconstrained equal sacrifice rules.

a constant measure of risk aversion must take one of two forms:

$$U_0(x_0) = \alpha_0 \ln(x_0) + \beta_0 \text{ where } \alpha_0 > 0,$$

or

$$U_0(x_0) = \alpha_0 x_0^{\rho_0} + \beta_0 \text{ where } \alpha_0 \rho_0 > 0.$$

Theorem 3. *The rule S satisfies Continuity, Consistency, Composition Down, PASM-Claims, and Homogeneity if and only if S is an equal sacrifice rule in which every agent has the same constant measure of relative risk aversion.*

Given our discussion of the claims monotonicity axioms, it should not be hard to see that strengthening PASM-Claims to Strict Claims Monotonicity in Theorem 1 will yield the unconstrained equal sacrifice rules, $\widehat{\mathcal{ES}}$.

Theorem 4. *The rule S satisfies Continuity, Consistency, Composition Down, and Strict Claims Monotonicity if and only if S is an unconstrained equal sacrifice rule.*

Our final result looks at what happens when we require the rule to treat agents equally.

Equal Treatment of Equals. For each problem (N, c, E) and each $\{i, j\} \subset N$, if $c_i = c_j$, then $S_i(N, c, E) = S_j(N, c, E)$.

This axiom imposes the requirement that individuals with the same pre-tax income will have the same post-tax income (which implies the same tax for these individuals). In conjunction with our other axioms, Equal Treatment of Equals characterizes the symmetric equal sacrifice rules, $\mathcal{ES}^*$.

Theorem 5. *The rule S satisfies Continuity, Consistency, Composition Down, PASM-Claims, and Equal Treatment of Equals if and only if S is a symmetric equal sacrifice rule.*

6. Related literature

The family of equal sacrifice rules is a special case of the family of parametric rules characterized by Stovall (2014a, Theorem 1). A parametric rule is defined by a continuous function $f : \mathbb{N} \times \mathbb{R}_{++} \times [a, b] \to \mathbb{R}_+$, where $-\infty \leq a < b \leq \infty$, such that (i) f is weakly increasing in the third argument, and (ii) for every $i \in \mathbb{N}$ and $c_i \in \mathbb{R}_{++}$ we have $f(i, c_i, a) = 0$ and $f(i, c_i, b) = c_i$. A parametric rule Par^f is defined as follows. For every (N, c, E) and for every $i \in N$,

$$Par_i^f(N, c, E) = f(i, c_i, \lambda),$$

where λ is chosen so that $\sum_N f(i, c_i, \lambda) = E$.[15] The axioms that characterize the family of parametric rules are Continuity, Consistency, Endowment Monotonicity, as well as two other technical axioms referred to as N-Continuity and Intrapersonal Consistency. Let $\mathcal{P}$ denote the family of parametric rules.

An important special case of this family is the family of symmetric parametric rules, originally characterized by Young (1987, Theorem 1). The axioms that characterize the family of symmetric parametric rules are Continuity, Consistency, and Equal Treatment of Equals. Let $\mathcal{P}^*$ denote the family of symmetric parametric rules.

It is easy to see that $\mathcal{ES}^* \subset \mathcal{P}^*$ by simply comparing the set of axioms characterizing each family. We show $\mathcal{ES} \subset \mathcal{P}$ using the definition of each family.[16] Let $a = -\infty$ and $b = 0$. Then for $U \in \mathcal{U}$ and for every $i \in \mathbb{N}$, define the parametric function

$$f(i, c_i, \lambda) \equiv \overline{U_i^{-1}}(U_i(c_i) + \lambda).$$

Naumova (2002, Theorem 2.1) provides a characterization of a family similar to $\widehat{\mathcal{ES}}$.[17] However, her definition of a problem is broader than the one we consider, allowing for the possibility of surplus sharing.[18] Because of this, her main axiom, called Path Independence, is stronger than Composition Down. Not only this, but Path Independence implies Strict Claims Monotonicity in this domain, though this is not explicitly shown by Naumova. Also, Strict Endowment Monotonicity is explicitly imposed. Thus Theorem 4 is analogous to, but distinct from, Naumova's result.

Chambers and Moreno-Ternero (2017, Theorem 1) characterize a family of rules which contains (but is broader than) $\mathcal{ES}^*$. Combining their nomenclature with ours, we will refer to the family they characterize as the generalized symmetric equal sacrifice rules, denoted $\mathcal{GES}^*$. The axioms

[15] This rule is well-defined because such a λ always exists, and if there are multiple such lambdas, the underlying allocation is the same for them.

[16] Thus it must be that the axioms in Theorem 1 imply the two technical axioms used by Stovall (2014a), N-Continuity and Intrapersonal Consistency.

[17] Naumova also includes the condition $\lim_{x\to\infty} U_i(x) = \infty$ in the definition of the family she characterizes.

[18] A surplus sharing problem is similar to a conflicting claims problem, but where $E \geq \sum_N c_i$.

that characterize this family of rules are Continuity, Consistency, Composition Down, and Equal Treatment of Equals. Given this characterization, it is easy to see that $\mathcal{ES}^* \subset \mathcal{GES}^* \subset \mathcal{P}^*$. Indeed, relative to their result, Theorem 5 only adds PASM-Claims, which shrinks the $\mathcal{GES}^*$ family dramatically.

As mentioned in section 3, Young (1988, Theorem 1) provides a characterization of $\widehat{\mathcal{ES}}^*$. The axioms he used are Continuity, Consistency, Composition Down, Strict Endowment Monotonicity, Equal Treatment of Equals, and one other axiom not yet introduced called Strict Order Preservation of Awards.

Strict Order Preservation of Awards. For each problem (N, c, E) where $E > 0$, and each $\{i, j\} \subset N$, if $c_i > c_j$, then $S_i(N, c, E) > S_j(N, c, E)$.

However, Theorem 4 and Theorem 5 together imply the following alternative characterization of the family $\widehat{\mathcal{ES}}^*$.

Corollary 1. *The rule S satisfies Continuity, Consistency, Composition Down, Strict Claims Monotonicity, and Equal Treatment of Equals if and only if S is an unconstrained and symmetric equal sacrifice rule.*

Corollary 1 is a tighter result than Young's. To see this, first note that in conjunction with the other axioms, Strict Order Preservation of Awards and Strict Claims Monotonicity are equivalent.[19] Thus the only difference between Corollary 1 and Young's result is the absence of Strict Endowment Monotonicity. Thus Corollary 1 demonstrates that Strict Endowment Monotonicity is implied by Young's other axioms.[20]

Table 1 summarizes this discussion by listing the axioms each of the above families of rules respectively satisfy. Figure 1 illustrates the logical relationships between these families. Returning to the examples given at the end of section 3, the proportional rule is a member of $\widehat{\mathcal{ES}}^*$, the constrained equal loss rule is a member of $\mathcal{ES}^*$, and the weighted constrained equal loss rule is a

[19] Specifically, we claim that if S satisfies Continuity, Consistency, and Equal Treatment of Equals, then S satisfies Strict Order Preservation of Awards if and only if S satisfies Strict Claims Monotonicity. This is easily proved by applying Young (1987, Theorem 1) to S.

[20] This can be proven directly by modifying the proof of Lemma 2 to get the following result: If S satisfies Consistency, Composition Down, and Strict Claims Monotonicity, then S satisfies Strict Endowment Monotonicity.

member of $\mathcal{ES}$. An example of a rule in $\widehat{\mathcal{ES}}$ would be a rule from Theorem 3 in which the measure of relative risk aversion is positive.

Table 1 and Figure 1 also raise the following open questions: What would be the asymmetric version of $\mathcal{GES}^*$? Would this family be characterized by Continuity, Consistency, and Composition Down? Or would N-Continuity and Intrapersonal Consistency need to assumed as well?

One other prominent result can be easily compared to our family of equal sacrifice rules. Moulin (2000, Theorem 2) characterizes the family of rules satisfying Consistency, Composition Down, the dual of Composition Down, and Homogeneity. The intersection of this family and the equal sacrifice family is the weighted constrained equal loss rules and the proportional rule. This observation demonstrates that adding PASM-Claims to Moulin's set of axioms substantially reduces his family of rules.

We conclude with the following observation. As pointed out by Young (1988, p.322), there are other ways to interpret the principle of equal sacrifice. For example, one may wish to instead equalize *marginal* sacrifice across individuals. However, since the marginal sacrifice of another dollar of taxation is identical to the marginal utility of another dollar of income, this equates to simply choosing post-tax income so as to maximize the sum of utilities. This is exactly the method of rules characterized by Stovall (2014b).

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	$\mathcal{P}$	$\mathcal{P}^*$	$\mathcal{GES}^*$	$\mathcal{ES}$	$\mathcal{ES}^*$	$\widehat{\mathcal{ES}}$	$\widehat{\mathcal{ES}}^*$
Continuity	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
N-Continuity	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
Consistency	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
Intrapersonal Consistency	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
Equal Treatment of Equals	$-$	$\oplus$	$\oplus$	$-$	$\oplus$	$-$	$\oplus$
Composition Down	$-$	$-$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
Claims Monotonicity	$-$	$-$	$+$	$+$	$+$	$+$	$+$
PASM-Claims	$-$	$-$	$-$	$\oplus$	$\oplus$	$+$	$+$
Strict Claims Monotonicity	$-$	$-$	$-$	$-$	$-$	$\oplus$	$\oplus$
Endowment Monotonicity	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
PASM-Endowment	$-$	$-$	$-$	$+$	$+$	$+$	$+$
Strict Endowment Monotonicity	$-$	$-$	$-$	$-$	$-$	$+$	$+$
Source	[1]	[2]	[3]	[4]	[5]	[6]	[7]

[1] Stovall (2014a, Theorem 1).

[2] Young (1987, Theorem 1).

[3] Chambers and Moreno-Ternero (2017, Theorem 1).

[4] Theorem 1.

[5] Theorem 5.

[6] Theorem 4, which is an analogue to Naumova (2002, Theorem 2.1).

[7] Corollary 1, which is a tighter result than Young (1988, Theorem 1).

Table 1: Summary of families of rules and axioms. *The symbols $+$ and $-$ indicate the axiom is necessary and not necessary, respectively. For any column, the set of axioms indicated by $\oplus$ are necessary and sufficient for the given family.*

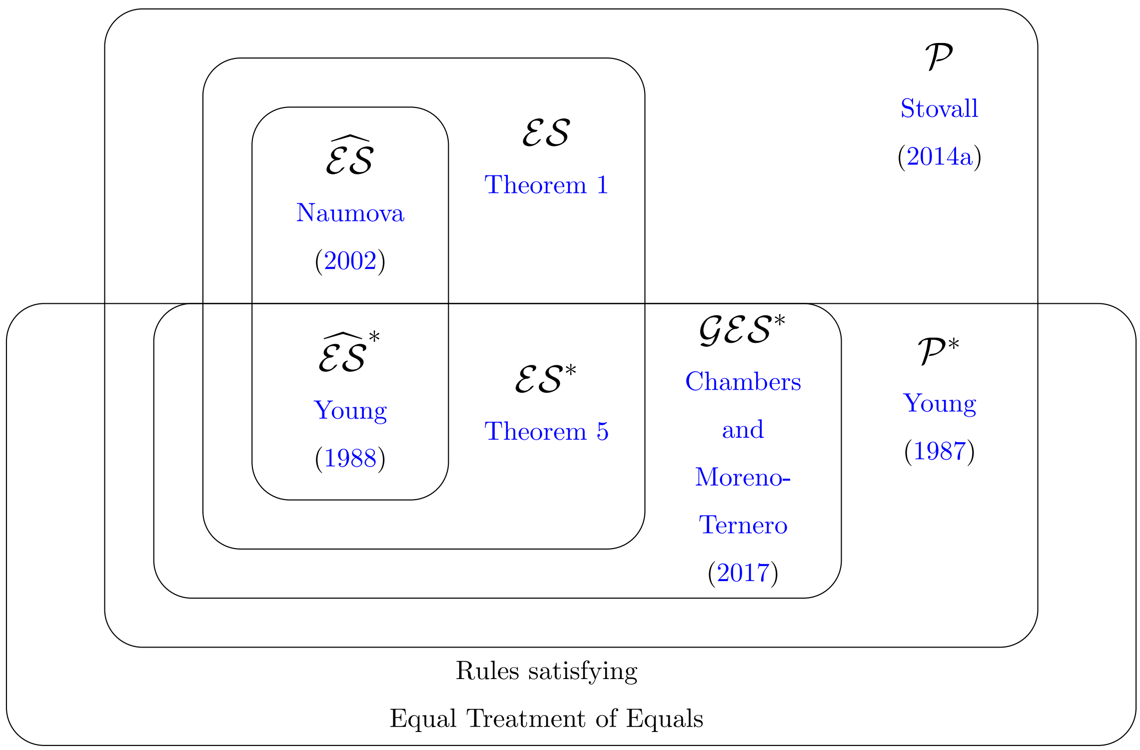

Figure 1: Diagram of logical relation among families of rules.

Appendix A. Notation

Let $\mathbb{Z}$ denote the set of integers. Let $\mathbb{D}$ denote the set of dyadic rationals, i.e.

$$\mathbb{D} \equiv \left\{ \frac{z}{2^{n-1}} : z \in \mathbb{Z}, n \in \mathbb{N} \right\}.$$

For an interval $[a, b]$ and $\mathbb{K} = \mathbb{N}, \mathbb{D}$, etc. let $\mathbb{K}[a, b]$ denote the set $\mathbb{K} \cap [a, b]$. Similar notation will be used for open or unbounded intervals.

For $a < b$ and $M \in \mathbb{N}$, we say $\{y^m\}_{m=0}^M$ is an *M-partition* of $[a, b]$ if

$$a = y^0 < y^1 < \ldots < y^{M-1} < y^M = b.$$

Appendix B. Proof of Proposition 1

The following lemma will be used for the proof. It is a special case of Aczél (1987, p. 20, Corollary 8) and gives the solutions to Cauchy's functional equation.

Lemma 1. *Suppose $I \subset \mathbb{R}$ satisfies either (1) $I = \mathbb{R}$ or (2) $I = (0, r)$ where $r \in \mathbb{R}_{++} \cup \{\infty\}$. If $f : I \to \mathbb{R}$ is continuous and satisfies*

$$f(x + y) = f(x) + f(y) \textit{ for all } x, y, x + y \in I,$$

then there exists $\alpha \in \mathbb{R}$ such that $f(x) = \alpha x$.

Now we prove Proposition 1.

Proof of Proposition 1. ($\Rightarrow$) Suppose $V \in \mathcal{U}$ is an equal sacrifice representation of ES^U. Fix $i \in \mathbb{N}$ and fix $j \neq i$. Set

$$M \equiv \min\left\{\sup\{U_i(y_i) - U_i(x_i) : y_i > x_i > 0\}, \sup\{U_j(y_j) - U_j(x_j) : y_j > x_j > 0\}\right\}.$$

Since both U and V are equal sacrifice representations of ES^U,

$$U_i(y_i) - U_i(x_i) = U_j(y_j) - U_j(x_j) \in (0, M) \Longrightarrow V_i(y_i) - V_i(x_i) = V_j(y_j) - V_j(x_j).$$

This implies

$$U_i(y_i) - U_i(x_i) = U_i(y_i') - U_i(x_i') \in (0, M) \Longrightarrow V_i(y_i) - V_i(x_i) = V_i(y_i') - V_i(x_i').$$

Thus there exists $F_i : (0, M) \to \mathbb{R}_{++}$ such that for every $y_i > x_i > 0$ we have

$$F_i\left(U_i(y_i) - U_i(x_i)\right) = V_i(y_i) - V_i(x_i).$$

Choose $m, m' > 0$ such that $m + m' < M$. Since $m + m' < M$, there exists $x_i, y_i \in \mathbb{R}_{++}$ such that $y_i > x_i$ and $U_i(y_i) - U_i(x_i) = m + m'$. Note that $0 < m < U_i(y_i) - U_i(x_i)$, which implies $U_i(x_i) < U_i(y_i) - m < U_i(y_i)$. Since U_i is continuous and strictly increasing, there exists $w_i \in (x_i, y_i)$ such that $U_i(w_i) = U_i(y_i) - m$. This then implies $U_i(y_i) - U_i(w_i) = m$ and $U_i(w_i) - U_i(x_i) = m'$. Thus

$$\begin{aligned}
F_i(m + m') &= F_i\left(U_i(y_i) - U_i(x_i)\right) \\
&= V_i(y_i) - V_i(x_i) \\
&= V_i(y_i) - V_i(w_i) + V_i(w_i) - V_i(x_i) \\
&= F_i\left(U_i(y_i) - U_i(w_i)\right) + F_i\left(U_i(w_i) - U_i(x_i)\right) \\
&= F_i\left(m\right) + F_i\left(m'\right).
\end{aligned}$$

Thus for every $m, m', m + m' \in (0, M)$, F_i satisfies Cauchy's equation. Also, since U_i and V_i are continuous, F_i must be continuous. By Lemma 1, there exists $\alpha_i \in \mathbb{R}$ such that $F_i(m) = \alpha_i m$. Thus

$$V_i(y_i) - V_i(x_i) = \alpha_i\left(U_i(y_i) - U_i(x_i)\right) \text{ for all } x_i, y_i \in \mathbb{R}_{++} \text{ satisfying } U_i(y_i) - U_i(x_i) < M.$$

Now fix any $x_i, y_i \in \mathbb{R}_{++}$ such that $x_i < y_i$. There exists $n \in \mathbb{N}$ such that

$$\ell \equiv \frac{U_i(y_i) - U_i(x_i)}{n} < M.$$

Set $w_i^0 = x_i$ and $w_i^n = y_i$. For $k = 1, 2, \ldots, n-1$, let $w_i^k \in \mathbb{R}_{++}$ be the unique number satisfying

$$U_i(w_i^k) = U_i(w_i^{k-1}) + \ell.$$

Then $\{w_i^k\}_{k=0}^n$ is an n-partition of $[x_i, y_i]$, and $U_i(w_i^k) - U_i(w_i^{k-1}) = \ell$ for every $k = 1, 2, \ldots, n$. Thus by the result above,

$$V_i(w_i^k) - V_i(w_i^{k-1}) = \alpha_i\left(U_i(w_i^k) - U_i(w_i^{k-1})\right) = \alpha_i \ell \text{ for every } k = 1, 2, \ldots, n.$$

Thus

$$\begin{aligned} V_i(y_i) - V_i(x_i) &= \sum_{k=1}^{n} V_i(w_i^k) - V_i(w_i^{k-1}) \\ &= n\alpha_i \ell \\ &= \alpha_i \left(U_i(y_i) - U_i(x_i)\right) \end{aligned}$$

for every $x_i, y_i \in \mathbb{R}_{++}$ such that $x_i < y_i$.

Now set $\beta_i \equiv V_i(1) - \alpha_i U_i(1)$. Then $V_i = \alpha_i U_i + \beta_i$. Note that $\alpha_i > 0$ since both U_i and V_i are strictly increasing functions.

Similarly, for any $j \in \mathbb{N}$, there exists $\alpha_j \in \mathbb{R}_{++}$ and $\beta_j \in \mathbb{R}$ such that $V_j = \alpha_j U_j + \beta_j$. But recall

$$U_i(y_i) - U_i(x_i) = U_j(y_j) - U_j(x_j) \Longrightarrow V_i(y_i) - V_i(x_i) = V_j(y_j) - V_j(x_j).$$

Since there exists $y_i > x_i > 0$ and $y_j > x_j > 0$ such that $U_i(y_i) - U_i(x_i) = U_j(y_j) - U_j(x_j)$, this implies

$$\alpha_i \left(U_i(y_i) - U_i(x_i)\right) = \alpha_j \left(U_j(y_j) - U_j(x_j)\right)$$

which implies $\alpha_i = \alpha_j$ for all $i, j \in \mathbb{N}$.

($\Leftarrow$) It is a straightforward exercise to show that V is an equal sacrifice representation of ES^U.

□

Appendix C. Proof of Theorem 1

Proving that the axioms are necessary is a straightforward exercise. Thus we only show that the axioms are sufficient to yield an equal sacrifice representation. So, let S satisfy Continuity, Consistency, Composition Down, and PASM-Claims. By the following lemma, S also satisfies PASM-Endowment.

Lemma 2. *If S satisfies Continuity, Consistency, Composition Down, and PASM-Claims, then S satisfies PASM-Endowment.*

Proof. By way of contradiction, suppose S does not satisfy PASM-Endowment. I.e. there exists (N, c, E), $E' \in (E, \sum_N c_j]$, and $i \in N$ such that, if $x \equiv S(N, c, E)$ and $x' \equiv S(N, c, E')$, then $x > \mathbf{0}$ and $x_i \geq x'_i$. Since $E' > E$, there exists $j \in N$ such that $x_j < x'_j$. By Consistency, $(x_i, x_j) = S(\{i, j\}, (c_i, c_j), x_i + x_j)$ and $(x'_i, x'_j) = S(\{i, j\}, (c_i, c_j), x'_i + x'_j)$. Note that $x_i + x_j \neq x'_i + x'_j$.

Case 1: $x_i + x_j < x'_i + x'_j$. Note $x'_i > 0$. By Continuity, we can assume $x'_j > 0$ without loss of generality. By Composition Down, $(x_i, x_j) = S(\{i,j\}, (x'_i, x'_j), x_i + x_j)$, which implies $x_i \le x'_i$. Hence, $x_i = x'_i$. But then we have $x_j = S_j(\{i,j\}, (x_i, x'_j), x_i + x_j)$ and $x_j = S_j(\{i,j\}, (x_i, x_j), x_i + x_j)$, which is a violation of PASM-Claims.

Case 2: $x_i + x_j > x'_i + x'_j$. By Composition Down, $(x'_i, x'_j) = S(\{i,j\}, (x_i, x_j), x'_i + x'_j)$, which implies $x'_j \le x_j$, which contradicts $x'_j > x_j$. □

Appendix C.1. Definitions and preliminary results

Define the set

$$Y \equiv \{(i, c_i, x_i) : i \in \mathbb{N}, 0 < x_i \le c_i\}$$

and its interior

$$Y^o \equiv \{(i, c_i, x_i) : i \in \mathbb{N}, 0 < x_i < c_i\}.$$

We call $(i, c_i, x_i) \in Y$ a *situation*. We think of a situation (i, c_i, x_i) as describing an agent i, her pre-tax income c_i, and her post-tax income x_i. Define the binary relation $\succsim$ over Y:[21]

$$(i, c_i, x_i) \succsim (j, c_j, x_j) \text{ if } S_i(\{i,j\}, (c_i, c_j), x_i + x_j) \ge x_i.$$

Let $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$ respectively. Note that $(i, c_i, x_i) \sim (j, c_j, x_j)$ if and only if $S(\{i,j\}, (c_i, c_j), x_i + x_j) = (x_i, x_j)$. In fact, Consistency implies the following result.

Lemma 3. *Suppose $x = S(N, c, E)$. Then for every $i, j \in N$ such that $x_i, x_j > 0$, we have $(i, c_i, x_i) \sim (j, c_j, x_j)$.*

The next three lemmas will be invoked often. We omit the proofs of the first two as they follow easily from Continuity, PASM-Endowment, and PASM-Claims.

Lemma 4. *Suppose $(i, c_i, x_i) \succ (j, c_j, x_j)$. Then $x_i < c_i$ and there exists a unique $x'_i \in (x_i, c_i]$ such that $(i, c_i, x'_i) \sim (j, c_j, x_j)$. Moreover, $x'_i = c_i$ if and only if $x_j = c_j$.*

Lemma 5. *Suppose $(i, c_i, x_i) \sim (j, c_j, x_j)$. Then*

[21] The binary relation defined here is similar to the one employed by Kaminski (2000) and Stovall (2014a). Kaminski (2000) also discusses the relation between this binary relation and the one used by Dagan and Volij (1997).

(i) $(i, c_i, x_i') \succ (j, c_j, x_j)$ *for every* $x_i' \in (0, x_i)$*;*

(ii) $(i, c_i, x_i) \succ (j, c_j, x_j')$ *for every* $x_j' \in (x_j, c_j]$*; and*

(iii) $(i, c_i, x_i) \succ (j, c_j', x_j)$ *for every* $c_j' \in [x_j, c_j)$*.*

Lemma 6. *Suppose* $(i, c_i, x_i) \sim (j, c_j, x_j)$*.*

(i) For every $x_j' \in [x_j, c_j]$*, there exists a unique* $\hat{x}_i(x_j') \in [x_i, c_i]$ *such that* $(i, c_i, \hat{x}_i(x_j')) \sim (j, c_j, x_j')$*. Moreover,* $\hat{x}_i(x_j')$ *is continuous and strictly increasing, with* $\hat{x}_i(x_j) = x_i$ *and* $\hat{x}_i(c_j) = c_i$*.*

(ii) For every $c_j' \in [x_j, c_j]$*, there exists a unique* $\hat{x}_i(c_j') \in [x_i, c_i]$ *such that* $(i, c_i, \hat{x}_i(c_j')) \sim (j, c_j', x_j)$*. Moreover,* $\hat{x}_i(c_j')$ *is continuous and strictly decreasing, with* $\hat{x}_i(x_j) = c_i$ *and* $\hat{x}_i(c_j) = x_i$*.*

Proof. (i) The existence of $\hat{x}_i(x_j')$ follows easily from Lemma 4 and item (ii) of Lemma 5. PASM-Endowment and Continuity imply that $\hat{x}_i(x_j')$ is strictly increasing and continuous.

(ii) The existence of $\hat{x}_i(c_j')$ follows easily from Lemma 4 and item (iii) of Lemma 5. PASM-Claims and Continuity imply that $\hat{x}_i(c_j')$ is strictly decreasing and continuous. □

The next two lemmas establish that $\sim$ is transitive.

Lemma 7. *Suppose* $(i, c_i, x_i) \sim (j, c_j, x_j) \sim (k, c_k, x_k)$*, where* $i \neq k$*. Then there exists* E *such that*

$$S(\{i, j, k\}, (c_i, c_j, c_k), E) = (x_i, x_j, x_k).$$

Proof. By Continuity, there exists E such that $S_i(\{i, j, k\}, (c_i, c_j, c_k), E) = x_i$. Let $x_j' = S_j(\{i, j, k\}, (c_i, c_j, c_k), E)$ and $x_k' = S_k(\{i, j, k\}, (c_i, c_j, c_k), E)$. Consistency then implies $(x_i, x_j') = S(\{i, j\}, (c_i, c_j), x_i + x_j')$, or $(i, c_i, x_i) \sim (j, c_j, x_j')$. Since $(i, c_i, x_i) \sim (j, c_j, x_j)$, item (i) of Lemma 6 then implies $x_j' = x_j$. Similarly, we can show $x_k' = x_k$. □

Lemma 8. *Suppose* $(i, c_i, x_i) \sim (j, c_j, x_j) \sim (k, c_k, x_k)$*, where* $i \neq k$*. Then* $(i, c_i, x_i) \sim (k, c_k, x_k)$*.*

Proof. This follows directly from Lemma 3 and Lemma 7. □

The final lemma in this subsection establishes the existence of what we think of as a 'halfway point' between a claims vector and its associated awards vector.

Lemma 9. *Let (N, x^1, E) be a problem where $|N| \geq 3$. Suppose $x^0 = S(N, x^1, E) > \mathbf{0}$. Then there exists a unique $x^{1/2}$ satisfying $x^0 < x^{1/2} < x^1$ such that for any $i, j \in N$ and $m, m' \in \{1, 2\}$, we have*

$$(i, x_i^{m/2}, x_i^{(m-1)/2}) \sim (j, x_j^{m'/2}, x_j^{(m'-1)/2}).$$

Proof. Fix $i, j \in N$. By Lemma 3, $(i, x_i^1, x_i^0) \sim (j, x_j^1, x_j^0)$. By item (i) of Lemma 6, there exists $\hat{x}_i(\cdot)$ continuous and strictly increasing such that for any $a \in [x_j^0, x_j^1]$, we have $(i, x_i^1, \hat{x}_i(a)) \sim (j, x_j^1, a)$. Note that when $a = x_j^0$, then $\hat{x}_i(a) = x_i^0$, so $(j, x_j^1, a) \succ (i, \hat{x}_i(a), x_i^0)$ by item (iii) of Lemma 5. Also, when $a = x_j^1$, then $\hat{x}_i(a) = x_i^1$, so $(i, \hat{x}_i(a), x_i^0) \succ (j, x_j^1, a)$ by part (ii) of Lemma 5.

Because S and $\hat{x}_i$ are continuous, there exists $x_j^{1/2} \in (x_j^0, x_j^1)$ such that $(j, x_j^1, x_j^{1/2}) \sim (i, \hat{x}_i(x_j^{1/2}), x_i^0)$. Set $x_i^{1/2} = \hat{x}_i(x_j^{1/2}) \in (x_i^0, x_i^1)$. Thus $(i, x_i^1, x_i^{1/2}) \sim (j, x_j^1, x_j^{1/2}) \sim (i, x_i^{1/2}, x_i^0)$. Since $(i, x_i^1, x_i^0) \sim (j, x_j^1, x_j^0)$ and $(i, x_i^1, x_i^{1/2}) \sim (j, x_j^1, x_j^{1/2})$, Composition Down implies $(i, x_i^{1/2}, x_i^0) \sim (j, x_j^{1/2}, x_j^0)$. Thus we have

$$(i, x_i^1, x_i^{1/2}) \sim (j, x_j^1, x_j^{1/2}) \sim (i, x_i^{1/2}, x_i^0) \sim (j, x_j^{1/2}, x_j^0).$$

Now fix $k \in N \setminus \{i, j\}$. By Lemma 3, $(k, x_k^1, x_k^0) \sim (j, x_j^1, x_j^0)$. By item (i) of Lemma 6, there exists $x_k^{1/2} = \hat{x}_k(x_j^{1/2}) \in (x_k^0, x_k^1)$ such that $(k, x_k^1, x_k^{1/2}) \sim (j, x_j^1, x_j^{1/2})$. Repeatedly applying Lemma 8 gives $(k, x_k^1, x_k^{1/2}) \sim (j, x_j^{1/2}, x_j^0)$ and $(k, x_k^1, x_k^{1/2}) \sim (i, x_i^1, x_i^{1/2})$. But then applying Lemma 8 one more time gives $(i, x_i^1, x_i^{1/2}) \sim (j, x_j^{1/2}, x_j^0)$. Thus we have

$$(i, x_i^1, x_i^{1/2}) \sim (j, x_j^1, x_j^{1/2}) \sim (i, x_i^{1/2}, x_i^0) \sim (j, x_j^{1/2}, x_j^0) \sim (i, x_i^1, x_i^{1/2}).$$

Indeed, similar reasoning yields

$$(k, x_k^1, x_k^{1/2}) \sim (j, x_j^1, x_j^{1/2}) \sim (k, x_k^{1/2}, x_k^0) \sim (j, x_j^{1/2}, x_j^0) \sim (k, x_k^1, x_k^{1/2}).$$

A similar process can show the above relations for any two agents in N. □

Appendix C.2. Measuring a situation

In this subsection, we establish a way of measuring a situation. Roughly, this is done by arbitrarily choosing three situations that are equivalent under $\sim$ to be the unit. For each of these 'units', the dyadic set is defined by recursively applying Lemma 9. This will allow us to measure situations that are 'less' than the unit. Thus the measure of a given situation will be the number of times the unit 'covers' the given situation.

Fix $x^1 \in \mathbb{R}^3_{++}$. Note that for every $E \in \left(\sum_{i=1}^3 x_i^1 - \min_i\{x_i^1\}, \sum_{i=1}^3 x_i^1\right)$ we have $S(\{1,2,3\}, x^1, E) > \mathbf{0}$. Thus PASM-Endowment implies that there exists E such that $x^0 \equiv S(\{1,2,3\}, x^1, E)$ satisfies $\mathbf{0} < x^0 < x^1$.

For $i \in \{1,2,3\}$, define the function $x_i : \mathbb{D}[0,1] \to [x_i^0, x_i^1]$ recursively as follows.[22] Set $x_i(0) = x_i^0$, $x_i(1) = x_i^1$. For $n = 1, 2, \ldots$ and $m \in \mathbb{N}[1, 2^{n-1}]$, let $x_i\left(\frac{2m-1}{2^n}\right) \in \left(x_i\left(\frac{2m-2}{2^n}\right), x_i\left(\frac{2m}{2^n}\right)\right)$ denote the unique numbers from Lemma 9 satisfying

$$\left(i, x_i\left(\frac{2m-m'}{2^n}\right), x_i\left(\frac{2m-m'-1}{2^n}\right)\right) \sim \left(j, x_j\left(\frac{2m-m''}{2^n}\right), x_j\left(\frac{2m-m''-1}{2^n}\right)\right) \tag{C.1}$$

for $j \in \{1,2,3\} \setminus i$ and $m', m'' \in \{0,1\}$.

Lemma 10. *For any $n \in \mathbb{N}[2,\infty)$, $m \in \mathbb{N}[2, 2^{n-1}]$, and $i, j \in \{1,2,3\}$,*

$$\left(i, x_i\left(\tfrac{2m-1}{2^n}\right), x_i\left(\tfrac{2m-2}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{2m-2}{2^n}\right), x_j\left(\tfrac{2m-3}{2^n}\right)\right).$$

Proof. Fix $n \in \mathbb{N}[2,\infty)$ and $m \in \mathbb{N}[2, 2^{n-1}]$. For any $i, j \in \{1,2,3\}$, (C.1) implies

$$\left(i, x_i\left(\tfrac{2m-1}{2^n}\right), x_i\left(\tfrac{2m-2}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{2m-1}{2^n}\right), x_j\left(\tfrac{2m-2}{2^n}\right)\right) \tag{C.2}$$

and

$$\left(i, x_i\left(\tfrac{2m-2}{2^n}\right), x_i\left(\tfrac{2m-3}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{2m-2}{2^n}\right), x_j\left(\tfrac{2m-3}{2^n}\right)\right). \tag{C.3}$$

Composition Down then implies

$$\left(i, x_i\left(\tfrac{2m-1}{2^n}\right), x_i\left(\tfrac{2m-3}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{2m-1}{2^n}\right), x_j\left(\tfrac{2m-3}{2^n}\right)\right).$$

Since this holds for all $i, j \in \{1,2,3\}$, Lemma 7 and Lemma 9 imply the existence of a unique half-point. But (C.2) and (C.3) then imply that this half-point must be $x_i(\frac{2m-2}{2^n})$ for $i \in \{1,2,3\}$. Thus Lemma 9 gives the desired result. $\square$

Lemma 11. *For any $n \in \mathbb{N}$, $m, m' \in \mathbb{N}[1, 2^n]$, and $i, j \in \{1,2,3\}$*

$$\left(i, x_i\left(\tfrac{m}{2^n}\right), x_i\left(\tfrac{m-1}{2^n}\right)\right) \sim \left(j, x_j\left(\frac{m'}{2^n}\right), x_j\left(\frac{m'-1}{2^n}\right)\right).$$

[22] The use of the group $\{1,2,3\}$ is arbitrary and does not materially affect the constructed utility functions, as Proposition 1 shows us. What is necessary is having a group of at least three agents so as to take full advantage of the implications of Consistency.

Proof. For $n = 1$, the result is true by (C.1). So assume $n \geq 2$. Without loss of generality, assume $m > m'$.

Case 1: m is odd, i.e. $m = 2\hat{m} - 1$ for some $\hat{m} \in \mathbb{N}[2, 2^{n-1}]$. Then by Lemma 10

$$\left(i, x_i\left(\tfrac{m}{2^n}\right), x_i\left(\tfrac{m-1}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{m-1}{2^n}\right), x_j\left(\tfrac{m-2}{2^n}\right)\right).$$

Equation (C.1) then implies

$$\left(j, x_j\left(\tfrac{m-1}{2^n}\right), x_j\left(\tfrac{m-2}{2^n}\right)\right) \sim \left(i, x_i\left(\tfrac{m-2}{2^n}\right), x_i\left(\tfrac{m-3}{2^n}\right)\right).$$

Repeatedly applying Lemma 10 and (C.1) yields a chain of relations $\sim$ from $\left(i, x_i\left(\frac{m}{2^n}\right), x_i\left(\frac{m-1}{2^n}\right)\right)$ to $\left(j, x_j\left(\frac{m'}{2^n}\right), x_j\left(\frac{m'-1}{2^n}\right)\right)$. Moreover, these relations hold for all $i, j \in \{1, 2, 3\}$. Repeated application of Lemma 8 then yields the desired result.

Case 2: m is even. The proof is similar to the first case, only applying (C.1) first and then Lemma 10 second. □

Lemma 12. *Let $d, d', \hat{d}, \hat{d}' \in \mathbb{D}[0, 1]$ satisfy $d - d' = \hat{d} - \hat{d}' > 0$. Then for any $i, j \in \{1, 2, 3\}$, we have*

$$(i, x_i(d), x_i(d')) \sim (j, x_j(\hat{d}), x_j(\hat{d}')).$$

Proof. Since $d, d', \hat{d}, \hat{d}' \in \mathbb{D}[0, 1]$ and $d - d' = \hat{d} - \hat{d}'$, there exists $n \in \mathbb{N}$, $m, \hat{m} \in \mathbb{N}[1, 2^n]$, and $\bar{m} \in \mathbb{N}[1, \min\{m, \hat{m}\}]$ such that $d = \frac{m}{2^n}$, $d' = \frac{m-\bar{m}}{2^n}$, $\hat{d} = \frac{\hat{m}}{2^n}$, and $\hat{d}' = \frac{\hat{m}-\bar{m}}{2^n}$.

By Lemma 11, we have

$$\left(i, x_i\left(\tfrac{m}{2^n}\right), x_i\left(\tfrac{m-1}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{\hat{m}}{2^n}\right), x_j\left(\tfrac{\hat{m}-1}{2^n}\right)\right)$$

for any $i, j \in \{1, 2, 3\}$. Similarly, we have

$$\left(i, x_i\left(\tfrac{m-1}{2^n}\right), x_i\left(\tfrac{m-2}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{\hat{m}-1}{2^n}\right), x_j\left(\tfrac{\hat{m}-2}{2^n}\right)\right).$$

Composition Down then implies

$$\left(i, x_i\left(\tfrac{m}{2^n}\right), x_i\left(\tfrac{m-2}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{\hat{m}}{2^n}\right), x_j\left(\tfrac{\hat{m}-2}{2^n}\right)\right).$$

Continuing in this way, we have

$$\left(i, x_i\left(\tfrac{m}{2^n}\right), x_i\left(\tfrac{m-\bar{m}}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{\hat{m}}{2^n}\right), x_j\left(\tfrac{\hat{m}-\bar{m}}{2^n}\right)\right),$$

or

$$(i, x_i(d), x_i(d')) \sim (j, x_j(\hat{d}), x_j(\hat{d}')),$$

as desired. ☐

Now for $i \in \{1, 2, 3\}$, we extend x_i from $\mathbb{D}[0, 1]$ to $[0, 1]$: For $a \in [0, 1]$, set

$$x_i(a) = \sup\{x_i(d) : d \in \mathbb{D}[0, 1] \text{ and } d \leq a\}.$$

The density of $\mathbb{D}$ in $\mathbb{R}$ in conjunction with Continuity and PASM-Endowment imply the following.

Lemma 13. *For $i \in \{1, 2, 3\}$, the function $x_i : [0, 1] \to [x_i^0, x_i^1]$ is continuous and strictly increasing.*

The following lemma follows easily from Lemma 12.

Lemma 14. *Let $a, b, \hat{a}, \hat{b} \in [0, 1]$ satisfy $b - a = \hat{b} - \hat{a} > 0$. Then for any $i, j \in \{1, 2, 3\}$, we have*

$$(i, x_i(b), x_i(a)) \sim (j, x_j(\hat{b}), x_j(\hat{a})).$$

For $i \in \{1, 2, 3\}$, let u_i denote the inverse of x_i. I.e. for $\hat{x}_i \in [x_i^0, x_i^1]$, we have $u_i(\hat{x}_i) = a$ if $x_i(a) = \hat{x}_i$. Lemma 13 then implies the following lemma.

Lemma 15. *For $i \in \{1, 2, 3\}$, the function $u_i : [x_i^0, x_i^1] \to [0, 1]$ is continuous and strictly increasing.*

Lemma 16. *Fix $i, j \in \{1, 2, 3\}$, $\hat{x}_i, \hat{x}'_i \in [x_i(0), x_i(1)]$, and $\hat{x}_j, \hat{x}'_j \in [x_j(0), x_j(1)]$. Then $(i, \hat{x}_i, \hat{x}'_i) \sim (j, \hat{x}_j, \hat{x}'_j)$ if and only if $u_i(\hat{x}_i) - u_i(\hat{x}'_i) = u_j(\hat{x}_j) - u_j(\hat{x}'_j)$.*

Proof. ($\Rightarrow$) By way of contradiction and without loss of generality, suppose $u_i(\hat{x}_i) - u_i(\hat{x}'_i) > u_j(\hat{x}_j) - u_j(\hat{x}'_j)$. Since u_i is continuous and strictly increasing by Lemma 15, there exists a unique $\hat{x}''_i \in (\hat{x}'_i, \hat{x}_i)$ such that $u_i(\hat{x}_i) - u_i(\hat{x}''_i) = u_j(\hat{x}_j) - u_j(\hat{x}'_j)$. Lemma 14 then implies

$$(i, x_i(u_i(\hat{x}_i)), x_i(u_i(\hat{x}''_i))) \sim (j, x_j(u_j(\hat{x}_j)), x_j(u_j(\hat{x}'_j))),$$

or

$$(i, \hat{x}_i, \hat{x}''_i) \sim (j, \hat{x}_j, \hat{x}'_j).$$

But since $\hat{x}''_i > \hat{x}'_i$, item (i) of Lemma 5 implies $(i, \hat{x}_i, \hat{x}'_i) \succ (j, \hat{x}_j, \hat{x}'_j)$, which is a contradiction.

($\Leftarrow$) This direction is a direct result of Lemma 14. ☐

Lemma 17. *Fix $i \in \{1,2,3\}$. Let $\hat{x}_i, \hat{x}'_i, \bar{x}_i, \bar{x}'_i \in [x_i(0), x_i(1)]$ and $(j, c_j, x_j) \in Y$ satisfy $j \neq i$ and*

$$(i, \hat{x}_i, \hat{x}'_i) \sim (j, c_j, x_j) \sim (i, \bar{x}_i, \bar{x}'_i).$$

Then $u_i(\hat{x}_i) - u_i(\hat{x}'_i) = u_i(\bar{x}_i) - u_i(\bar{x}'_i)$.

Proof. Choose $k \in \{1,2,3\} \setminus \{i,j\}$. Since u_k is strictly increasing and continuous by Lemma 15, there exists a unique $\hat{x}_k \in [x_k(0), x_k(1)]$ such that $u_k(\hat{x}_k) = u_i(\hat{x}_i) - u_i(\hat{x}'_i)$. Since $u_k(x_k(0)) = 0$, this means $u_k(\hat{x}_k) - u_k(x_k(0)) = u_i(\hat{x}_i) - u_i(\hat{x}'_i)$, so by Lemma 16 we have $(k, \hat{x}_k, x_k(0)) \sim (i, \hat{x}_i, \hat{x}'_i)$. Lemma 8 applied twice implies $(k, \hat{x}_k, x_k(0)) \sim (i, \bar{x}_i, \bar{x}'_i)$. Lemma 16 implies $u_k(\hat{x}_k) - u_k(x_k(0)) = u_i(\bar{x}_i) - u_i(\bar{x}'_i)$. Thus $u_i(\hat{x}_i) - u_i(\hat{x}'_i) = u_i(\bar{x}_i) - u_i(\bar{x}'_i)$. □

The next lemma will be a key part in establishing a measure of a situation.

Lemma 18. *For any $(i, c_i, x_i) \in Y^{\circ}$ and $j \in \{1,2,3\} \setminus \{i\}$, there exist unique*

(i) $\{y^m\}_{m=0}^{M}$ an M-partition of $[x_i, c_i]$, and

(ii) $\ell \in [x_j(0), x_j(1))$

such that $(i, y^m, y^{m-1}) \sim (j, x_j(1), x_j(0))$ for every $m \in \{2, \ldots, M\}$, and $(i, y^1, y^0) \sim (j, x_j(1), \ell)$.

Proof. Fix $(i, c_i, x_i) \in Y^{\circ}$ and $j \in \{1,2,3\} \setminus \{i\}$. Define the sequence $\{\hat{y}^m\}$ recursively: Set $\hat{y}^0 = c_i$. If $(i, \hat{y}^{m-1}, x_i) \succ (j, x_j(1), x_j(0))$, then by Lemma 4 there exists a unique $\bar{y} \in (x_i, \hat{y}^{m-1})$ such that $(i, \hat{y}^{m-1}, \bar{y}) \sim (j, x_j(1), x_j(0))$. (Note that $\bar{y} \neq \hat{y}^{m-1}$ since $x_1(1) \neq x_1(0)$.) Set $\hat{y}^m = \bar{y}$ so that $(i, \hat{y}^{m-1}, \hat{y}^m) \sim (j, x_j(1), x_j(0))$. If $(j, x_j(1), x_j(0)) \succsim (i, \hat{y}^{m-1}, x_i)$, then set $\hat{y}^m = x_i$ and $M = m$. The following claim shows that this case will happen for finite m.

Claim: There exists m such that $(j, x_j(1), x_j(0)) \succsim (i, \hat{y}^{m-1}, x_i)$. By way of contradiction, suppose not, i.e. $(i, \hat{y}^m, x_i) \succ (j, x_j(1), x_j(0))$ for all $m \in \mathbb{N}$. Since $\{\hat{y}^m\}_{\mathbb{N}}$ is a strictly decreasing sequence with a lower bound x_i, it converges. Let $\hat{y}^m \to \tilde{y} \geq x_i$. Consider the sequence of problems $\{(\{i,j\}, (\hat{y}^{m-1}, x_j(1)), \hat{y}^m + x_j(0))\}_{\mathbb{N}}$. Since $(i, \hat{y}^{m-1}, \hat{y}^m) \sim (j, x_j(1), x_j(0))$ for every $m \in \mathbb{N}$, this means $S(\{i,j\}, (\hat{y}^{m-1}, x_j(1)), \hat{y}^m + x_j(0)) = (\hat{y}^m, x_j(0))$ for every $m \in \mathbb{N}$. Thus

$$S(\{i,j\}, (\hat{y}^{m-1}, x_j(1)), \hat{y}^m + x_j(0)) \to (\tilde{y}, x_j(0)).$$

However Continuity implies

$$S(\{i,j\}, (\hat{y}^{m-1}, x_j(1)), \hat{y}^m + x_j(0)) \to S(\{i,j\}, (\tilde{y}, x_j(1)), \tilde{y} + x_j(0)).$$

Thus $S(\{i,j\},(\tilde{y},x_j(1)),\tilde{y}+x_j(0))=(\tilde{y},x_j(0))$. But since $x_j(1)>x_j(0)>0$ and $\tilde{y}\geq x_i>0$, this would violate PASM-Endowment. This proves the claim.

To determine ℓ, there are two sub-cases to consider: If $(j,x_j(1),x_j(0))\sim(i,\hat{y}^{M-1},x_i)$, then set $\ell=x_j(0)$. If $(j,x_j(1),x_j(0))\succ(i,\hat{y}^{M-1},x_i)$, then by Lemma 4 there exists a unique $\ell\in(x_j(0),x_j(1))$ such that $(j,x_j(1),\ell)\sim(i,\hat{y}^{M-1},x_i)$. (Note that $\ell\neq x_j(1)$ since $\hat{y}^{M-1}\neq x_i$.)

For every $m\in\{0,1,2,\ldots,M\}$, set $y^m=\hat{y}^{M-m}$. Then $\{y^m\}_{m=0}^M$ and ℓ satisfy the desired requirements. $\square$

For any $(i,c_i,x_i)\in Y^\circ$ and $j\in\{1,2,3\}\backslash\{i\}$, let $M_j(i,c_i,x_i)$ denote the M and $\ell_j(i,c_i,x_i)$ denote the ℓ from Lemma 18. The next lemma shows that the choice of j is without loss of generality.

Lemma 19. *For any $(i,c_i,x_i)\in Y^\circ$ and $j,k\in\{1,2,3\}\setminus\{i\}$, we have $M_j(i,c_i,x_i)=M_k(i,c_i,x_i)$ and $u_j(\ell_j(i,c_i,x_i))=u_k(\ell_k(i,c_i,x_i))$.*

Proof. Abusing notation, let $M_j=M_j(i,c_i,x_i)$, $M_k=M_k(i,c_i,x_i)$, $\ell_j=\ell_j(i,c_i,x_i)$, and $\ell_k=\ell_k(i,c_i,x_i)$. Let $\{y_j^m\}_{m=0}^{M_j}$ and $\{y_k^m\}_{m=0}^{M_k}$ be the respective partitions of $[x_i,c_i]$ from Lemma 18. Since $(j,x_j(1),x_j(0))\sim(k,x_k(1),x_k(0))$, Lemma 8 implies that we must have $M_j=M_k$ and $y_j^m=y_k^m$ for all m.

Simplifying notation, let $\{y^m\}_{m=0}^M$ denote the partition of $[x_i,c_i]$ now. Since $(i,y^1,y^0)\sim(j,x_j(1),\ell_j)$ and $(i,y^1,y^0)\sim(k,x_k(1),\ell_k)$, Lemma 18 implies $(j,x_j(1),\ell_j)\sim(k,x_k(1),\ell_k)$. Lemma 16 then implies $u_j(x_j(1))-u_j(\ell_j)=u_k(x_k(1))-u_k(\ell_k)$. But $u_j(x_j(1))=u_k(x_k(1))=1$. Hence $u_j(\ell_j)=u_k(\ell_k)$. $\square$

Our measure of a situation (i,c_i,x_i) is given by $M_j(i,c_i,x_i)-u_j(\ell_j(i,c_i,x_i))$, where $j\in\{1,2,3\}\setminus\{i\}$. The previous lemma shows this measure is independent of j. The final lemma of this subsection shows that this measure is additive.

Lemma 20. *Suppose $i\in\mathbb{N}$ and $c>b>a>0$. Then for any $j\in\{1,2,3\}\setminus\{i\}$, we have*

$$M_j(i,c,b)-u_j(\ell_j(i,c,b))+M_j(i,b,a)-u_j(\ell_j(i,b,a))=M_j(i,c,a)-u_j(\ell_j(i,c,a)).$$

Proof. To simplify the proof, we will assume $1=M_j(i,c,b)=M_j(i,b,a)$. Generalizing the proof is a straightforward but tedious exercise.

Set $\ell'=\ell_j(i,c,b)$ and $\ell''=\ell_j(i,b,a)$. Thus we have $(i,c,b)\sim(j,x_j(1),\ell')$ and $(i,b,a)\sim(j,x_j(1),\ell'')$.

Case 1: $(j, x_j(1), x_j(0)) \succsim (i, c, a)$. If $(j, x_j(1), x_j(0)) \sim (i, c, a)$, then set $\hat{\ell} = x_j(0)$. Otherwise, if $(j, x_j(1), x_j(0)) \succ (i, c, a)$, then by Lemma 4 there exists a unique $\hat{\ell} \in (x_j(0), x_j(1))$ satisfying $(j, x_j(1), \hat{\ell}) \sim (i, c, a)$. Thus $M_j(i, c, a) = 1$ and $\ell_j(i, c, a) = \hat{\ell}$. Also, since $(i, c, b) \sim (j, x_j(1), \ell')$, Composition Down implies $(i, b, a) \sim (j, \ell', \hat{\ell})$. Since $(i, b, a) \sim (j, x_j(1), \ell'')$ by assumption, we have

$$(j, x_j(1), \ell'') \sim (i, b, a) \sim (j, \ell', \hat{\ell}).$$

Lemma 17 then implies $u_j(x_j(1)) - u_j(\ell'') = u_j(\ell') - u_j(\hat{\ell})$, or

$$u_j(\hat{\ell}) = u_j(\ell'') + u_j(\ell') - 1.$$

Thus we have:

$$\begin{aligned} M_j(i, c, a) - u_j(\ell_j(i, c, a)) &= 1 - u_j(\hat{\ell}) \\ &= 2 - u_j(\ell') - u_j(\ell'') \\ &= [1 - u_j(\ell')] + [1 - u_j(\ell'')] \\ &= [M_j(i, c, b) - u_j(\ell_j(i, c, b))] + [M_j(i, b, a) - u_j(\ell_j(i, b, a))]. \end{aligned}$$

Case 2: $(i, c, a) \succ (j, x_j(1), x_j(0))$. By Lemma 4, there exists a unique $y \in (a, c)$ such that $(i, c, y) \sim (j, x_j(1), x_j(0))$. (Note that $y < c$ since $x_j(1) > x_j(0)$.) In fact, it must be that $y \leq b$ since $M_j(i, c, b) = 1$.

Case 2(a): $y = b$. Then we must have $\ell' = x_j(0)$ which implies $u_j(\ell') = 0$. Thus

$$M_j(i, c, b) - u_j(\ell_j(i, c, b)) + M_j(i, b, a) - u_j(\ell_j(i, b, a)) = 2 - u_j(\ell'').$$

Also, $\ell' = x_j(0)$ implies $(i, c, b) \sim (j, x_j(1), x_j(0))$. But since $(i, b, a) \sim (j, x_j(1), \ell'')$, this implies $M_j(i, c, a) = 2$ and $\ell_j(i, c, a) = \ell''$. Thus

$$M_j(i, c, a) - u_j(\ell_j(i, c, a)) = 2 - u(\ell'')$$

as desired.

Case 2(b): $y < b$. Then we must have $\ell' > x_j(0)$. Since $(i, c, y) \sim (j, x_j(1), x_j(0))$ and $(i, c, b) \sim (j, x_j(1), \ell')$, Composition Down implies $(i, b, y) \sim (j, \ell', x_j(0))$. Since $(j, x_j(1), \ell'') \sim (i, b, a)$ and $y > a$, item (i) of Lemma 6 implies there exists a unique $\hat{\ell} \in (\ell'', x_j(1))$ such that $(j, x_j(1), \hat{\ell}) \sim (i, b, y)$. Thus we have

$$(j, x_j(1), \hat{\ell}) \sim (i, b, y) \sim (j, \ell', x_j(0)).$$

Lemma 17 then implies $u_j(x_j(1)) - u_j(\hat{\ell}) = u_j(\ell') - u_j(x_j(0))$, or

$$1 - u_j(\hat{\ell}) = u_j(\ell'). \tag{C.4}$$

Furthermore, since $(j, x_j(1), \ell'') \sim (i, b, a)$ and $(j, x_j(1), \hat{\ell}) \sim (i, b, y)$, Composition Down implies $(j, \hat{\ell}, \ell'') \sim (i, y, a)$. Since $(j, x_j(1), \ell'') \sim (i, b, a)$ and $y < b$, item (ii) of Lemma 6 implies there exists a unique $\hat{\ell}' \in (\ell'', x_j(1))$ such that $(j, x_j(1), \hat{\ell}') \sim (i, y, a)$. Thus we have

$$(j, x_j(1), \hat{\ell}') \sim (i, y, a) \sim (j, \hat{\ell}, \ell'').$$

Lemma 17 then implies $u_j(x_j(1)) - u_j(\hat{\ell}') = u_j(\hat{\ell}) - u_j(\ell'')$, or

$$1 - u_j(\hat{\ell}') = u_j(\hat{\ell}) - u_j(\ell''). \tag{C.5}$$

To summarize, we have $(i, c, y) \sim (j, x_j(1), x_j(0))$, $(i, y, a) \sim (j, x_j(1), \hat{\ell}')$. Thus $M_j(i, c, a) = 2$ and $\ell_j(i, c, a) = \hat{\ell}'$. Using this, (C.4), and (C.5), we get:

$$\begin{aligned} M_j(i,c,a) - u_j(\ell_j(i,c,a)) &= 2 - u_j(\hat{\ell}') \\ &= 2 - u_j(\ell') - u_j(\ell'') \\ &= [1 - u_j(\ell')] + [1 - u_j(\ell'')] \\ &= [M_j(i,c,b) - u_j(\ell(i,c,b))] + [M_j(i,b,a) - u_j(\ell(i,b,a))]. \end{aligned}$$

□

Appendix C.3. Defining utilities and finishing the proof

For any $i \in \mathbb{N}$ and $x_i > 0$, define

$$U_i(x_i) = \begin{cases} M_j(i, x_i, 1) - u_j\left(\ell_j(i, x_i, 1)\right) & \text{if } x_i > 1 \\ 0 & \text{if } x_i = 1 \\ u_j\left(\ell_j(i, 1, x_i)\right) - M_j(i, 1, x_i) & \text{if } x_i < 1, \end{cases}$$

where $j \in \{1, 2, 3\} \setminus \{i\}$. Lemma 19 implies that U_i is independent of j. Lemma 20 implies the following lemma.

Lemma 21. *For any $i \in \mathbb{N}$, $x_i > x_i' > 0$, and $j \in \{1, 2, 3\} \setminus \{i\}$, we have $U_i(x_i) - U_i(x_i') = M_j(i, x_i, x_i') - u_j(\ell_j(i, x_i, x_i'))$.*

The final four lemmas establish that $\{U_i\}_{i \in \mathbb{N}}$ satisfies all the conditions to be an equal sacrifice representation of S.

Lemma 22. *Suppose $(i, c_i, x_i), (j, c_j, x_j) \in Y$ and $i \neq j$. If $(i, c_i, x_i) \sim (j, c_j, x_j)$, then $U_i(c_i) - U_i(x_i) = U_j(c_j) - U_j(x_j)$.*

Proof. PASM-Endowment implies $c_i = x_i$ if and only if $c_j = x_j$. But in that case we would have $U_i(c_i) - U_i(x_i) = U_j(c_j) - U_j(x_j) = 0$.

Now suppose $c_i > x_i$ and $c_j > x_j$. Choose $k \in \{1, 2, 3\} \setminus \{i, j\}$. Set $\hat{M} = M_k(i, c_i, x_i)$ and $\hat{\ell} = \ell_k(i, c_i, x_i)$. Let $\{y^m\}_{m=0}^{\hat{M}}$ be the $\hat{M}$-partition of $[x_i, c_i]$ such that $(i, y^m, y^{m-1}) \sim (k, x_k(1), x_k(0))$ for every $m \in \{2, \ldots, \hat{M}\}$, and $(i, y^1, y^0) \sim (k, x_k(1), \hat{\ell})$. Define $z^0 = x_j$ and $z^{\hat{M}} = c_j$. For $m \in \{1, 2, \ldots \hat{M} - 1\}$, define z^m to be the unique award for j satisfying $(j, c_j, z^m) \sim (i, c_i, y^m)$. (Item (i) of Lemma 6 shows $\{z^m\}_{m=0}^{\hat{M}}$ is strictly increasing and unique since $(j, c_j, x_j) \sim (i, c_i, x_i)$ and $\{y^m\}_{m=0}^{\hat{M}}$ is strictly increasing.) Composition Down implies $(j, z^m, z^{m-1}) \sim (i, y^m, y^{m-1})$ for every $m \in \{1, 2, \ldots, M\}$. Lemma 8 then implies $(j, z^m, z^{m-1}) \sim (k, x_k(1), x_k(0))$ for every $m \in \{2, 3, \ldots, M\}$, and $(j, z^1, z^0) \sim (k, x_k(1), \hat{\ell})$. Hence $M(j, c_j, x_j) = \hat{M}$ and $\ell(j, c_j, x_j) = \hat{\ell}$. Lemma 21 then implies $U_i(c_i) - U_i(x_i) = \hat{M} - u_k(\hat{\ell}) = U_j(c_j) - U_j(x_j)$. $\square$

Lemma 23. *Suppose $x = S(N, c, E)$. Then for every $i, j \in N$ such that $x_i, x_j > 0$, we have $U_i(c_i) - U_i(x_i) = U_j(c_j) - U_j(x_j)$.*

Proof. This is follows directly from Lemma 3 and Lemma 22. $\square$

Lemma 24. *For every $i \in \mathbb{N}$, the function U_i is strictly increasing.*

Proof. By Lemma 21, U_i is strictly increasing if $M_j(i, x_i, x_i') - u_j(\ell_j(i, x_i, x_i')) > 0$ when $x_i > x_i' > 0$ and $j \in \{1, 2, 3\} \setminus \{i\}$. Note that $\ell_j(i, x_i, x_i') < x_j(1)$, which implies $u_j(\ell_j(i, x_i, x_i')) < 1$. Also $M_j(i, x_i, x_i') \geq 1$. Hence we must have $M_j(i, x_i, x_i') - u_j(\ell_j(i, x_i, x_i')) > 0$. $\square$

Lemma 25. *For every $i \in \mathbb{N}$, the function U_i is continuous.*

Proof. This follows easily from Continuity and Lemma 23. $\square$

Thus U_i is continuous and strictly increasing for every i, and $\{U_i\}_{i \in \mathbb{N}}$ is an equal sacrifice representation of S.

Appendix D. Proof of Theorem 2

First we state a standard result for concave functions.

Lemma 26. *A function $f : A \to \mathbb{R}$ is concave if and only if $f(a+h) - f(a) \geq f(b+h) - f(b)$ for $a < b$ and $h > 0$.*

The following lemma will be used in the proof.

Lemma 27. *Let $f : A \to \mathbb{R}$ be continuous. Suppose there exists $a < b$ and $\alpha \in (0,1)$ such that $(1-\alpha)f(a) + \alpha f(b) > f((1-\alpha)a + \alpha b)$. Then there exists $x \in (a,b)$ such that for every ϵ satisfying $0 < \epsilon < \min\{x-a, b-x\}$, we have $f(x) - f(x-\epsilon) < f(x+\epsilon) - f(x)$.*

Proof. Define the function $g : [0,1] \to \mathbb{R}$ to be

$$g(\beta) \equiv f((1-\beta)a + \beta b) - [(1-\beta)f(a) + \beta f(b)]\,.$$

Note that $g(0) = g(1) = 0$, $g(\alpha) < 0$, and g is continuous. By the Extreme Value Theorem, g attains a global minimum on $[0,1]$. Define

$$\gamma \equiv \min\{\beta \in [0,1] : g(\beta) \leq g(\beta') \text{ for all } \beta' \in [0,1]\}.$$

Note that $\gamma \in (0,1)$ since $g(\alpha) < 0 = g(0) = g(1)$ and $\alpha \in (0,1)$.

Set

$$x \equiv (1-\gamma)a + \gamma b.$$

Note that $x \in (a,b)$. Now choose ϵ satisfying $0 < \epsilon < \min\{x-a, b-x\}$. Set $\beta' \equiv \frac{x-\epsilon-a}{b-a} = \gamma - \frac{\epsilon}{b-a}$ and $\beta'' \equiv \frac{x+\epsilon-a}{b-a} = \gamma + \frac{\epsilon}{b-a}$. Note then that $0 < \beta' < \gamma < \beta'' < 1$. Since γ is a global minimum of g, we have $g(\beta') > g(\gamma)$ and $g(\beta'') \geq g(\gamma)$. The first inequality implies

$$\begin{aligned} f(x-\epsilon) - [(1-\beta')f(a) + \beta' f(b)] &> f(x) - [(1-\gamma)f(a) + \gamma f(b)] \\ (\gamma - \beta')\,[f(b) - f(a)] &> f(x) - f(x-\epsilon), \end{aligned}$$

while the second inequality implies

$$\begin{aligned} f(x+\epsilon) - [(1-\beta'')f(a) + \beta'' f(b)] &\geq f(x) - [(1-\gamma)f(a) + \gamma f(b)] \\ f(x+\epsilon) - f(x) &\geq (\beta'' - \gamma)\,[f(b) - f(a)]\,. \end{aligned}$$

But since $\gamma - \beta' = \frac{\epsilon}{b-a} = \beta'' - \gamma$, this implies

$$f(x+\epsilon) - f(x) \geq \frac{\epsilon}{b-a}\left[f(b) - f(a)\right] > f(x) - f(x-\epsilon),$$

as desired. □

Now we turn to the proof of Theorem 2.

($\Leftarrow$) By assumption, S is an equal sacrifice rule with representation U, where U_i is concave for every i. By Theorem 1, we only need to show that S satisfies Bounded Gain from Linked Claim-Endowment Increase.

Fix the problem (N, c, E), $i \in N$, and $h > 0$. Set $x \equiv S(N, c, E)$ and $x' \equiv S(N, (c_i + h, c_{-i}), E + h)$. By way of contradiction, suppose $h < x'_i - x_i$. Since $x_i + h < x'_i$, this implies that there exists $j \in N \setminus \{i\}$ such that $x'_j < x_j$. Consistency implies $(x_i, x_j) = S(\{i, j\}, (c_i, c_j), x_i + x_j)$ and $(x'_i, x'_j) = S(\{i, j\}, (c_i + h, c_j), x'_i + x'_j)$. Since $x'_i > 0$, $x_j > 0$, and U is an equal sacrifice representation of S, we must have

$$U_i(c_i + h) - U_i(x'_i) \geq U_j(c_j) - U_j(x'_j)$$

and

$$U_j(c_j) - U_j(x_j) \geq U_i(c_i) - U_i(x_i).$$

Also, since U_i is concave and $h > 0$, Lemma 26 implies

$$U_i(c_i) - U_i(x_i) \geq U_i(c_i + h) - U_i(x_i + h).$$

Finally, since U_i is strictly increasing, we have

$$U_i(c_i + h) - U_i(x_i + h) > U_i(c_i + h) - U_i(x'_i).$$

Putting this all together, we have

$$U_j(c_j) - U_j(x_j) > U_j(c_j) - U_j(x'_j).$$

But this implies $U_j(x_j) < U_j(x'_j)$, or $x_j < x'_j$ since U_j is strictly increasing. This contradicts $x'_j < x_j$.

($\Rightarrow$) Let S satisfy the stated axioms. By Theorem 1, there exists $U \in \mathcal{U}$ such that U is an equal sacrifice representation of S. By way of contradiction, suppose there exists i such that U_i is not concave. I.e. there exists $b > a > 0$ and $\alpha \in (0,1)$ such that $(1-\alpha)U_i(a)+\alpha U_i(b) > U_i((1-\alpha)a+\alpha b)$. By Lemma 27, there exists $c_i \in (a,b)$ such that for every δ satisfying $0 < \delta < \min\{c_i - a, b - c_i\}$, we have $U_i(c_i) - U_i(c_i - \delta) < U_i(c_i + \delta) - U_i(c_i)$. Choose $h' < \min\{c_i - a, b - c_i\}$.

Fix $j \neq i$ and $c_j > 0$. Choose $\epsilon < \min\{U_i(c_i) - U_i(c_i - h'), \lim_{a\to 0} U_j(c_j) - U_j(a)\}$. Since U_i and U_j are continuous and strictly increasing, there exists $x_i \in (0, c_i)$ and $x_j \in (0, c_j)$ such that $U_j(c_j) - U_j(x_j) = U_i(c_i) - U_i(x_i) = \epsilon$. Note that this implies $S(\{i,j\},(c_i,c_j),x_i + x_j) = (x_i, x_j)$. Set $h \equiv c_i - x_i$. Thus we have $x_i = c_i - h$ and $x_i, c_i + h \in (a,b)$. By Lemma 27, $U_i(c_i) - U_i(x_i) < U_i(c_i+h) - U_i(c_i)$. But then this implies $U_i(c_i+h) - U_i(c_i) > U_j(c_j) - U_j(x_j)$. Since U_i and U_j are both continuous and strictly increasing, this means that $S_i(\{i,j\},(c_i+h,c_j),c_i+x_j) > c_i = x_i + h$. But this implies $S_i(\{i,j\},(c_i+h,c_j),c_i+x_j) - S_i(\{i,j\},(c_i,c_j),x_i+x_j) > h$, which violates Bounded Gain from Linked Claim-Endowment Increase.

Appendix E. Proof of Theorem 3

First we state two lemmas which are straightforward corollaries of Lemma 1.

Lemma 28. *If $f : \mathbb{R}_{++} \to \mathbb{R}$ is continuous and satisfies*

$$f(xy) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}_{++},$$

then there exists $\alpha \in \mathbb{R}$ such that $f(x) = \alpha \ln(x)$.

This can be easily proven by applying Lemma 1 to $g(x) \equiv f(e^x)$.

Lemma 29. *If $f : \mathbb{R}_{++} \to \mathbb{R}$ is continuous and satisfies*

$$f(xy) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}_{++},$$

then either $f(x) = 0$ or there exists $\alpha \in \mathbb{R}$ such that $f(x) = x^\alpha$.

This can be easily proven by applying Lemma 28 to $g(x) \equiv \ln(f(x))$. We turn now to the proof of Theorem 3.

It is a straightforward exercise to show that Homogeneity is necessary. Therefore we only show the sufficiency part of the proof. In particular, we show that if S satisfies the stated axioms, then either:

1. For every $i \in \mathbb{N}$, $U_i(x_i) = \alpha_i \ln(x_i) + \beta_i$ where $\alpha_i > 0$; or

2. There exists $\rho \neq 0$ such that for every $i \in \mathbb{N}$, $U_i(x_i) = \alpha_i x_i^{\rho} + \beta_i$ where $\alpha_i \rho > 0$.

By Theorem 1, there exists $U \in \mathcal{U}$ such that U is an equal sacrifice representation of S. By Proposition 1, $\hat{U} \in \mathcal{U}$ defined as

$$\hat{U}_i(x_i) \equiv U_i(x_i) - U_i(1) \text{ for all } i \in \mathbb{N}$$

is also an equal sacrifice representation of S. Note that $\hat{U}_i(1) = 0$ for all i. For any $\lambda > 0$, define $V^{\lambda} \in \mathcal{U}$ as

$$V_i^{\lambda}(x_i) \equiv \hat{U}_i(\lambda x_i) \text{ for all } i \in \mathbb{N}.$$

Note that for fixed λ, V^{λ} is an equal sacrifice representation of S. To see this, suppose $y_i > x_i > 0$ and $y_j > x_j > 0$ satisfy $\hat{U}(y_i) - \hat{U}_i(x_i) = \hat{U}_j(y_j) - \hat{U}_j(x_j)$. By Homogeneity, we must have $\hat{U}(\lambda y_i) - \hat{U}_i(\lambda x_i) = \hat{U}_j(\lambda y_j) - \hat{U}_j(\lambda x_j)$. But then $V_i^{\lambda}(y_i) - V_i^{\lambda}(x_i) = V_j^{\lambda}(y_j) - V_j^{\lambda}(x_j)$ by definition.

Therefore by Proposition 1, there exist $\alpha(\lambda) \in \mathbb{R}_{++}$ and $\beta(\lambda) \in \mathbb{R}^{\mathbb{N}}$ such that

$$V_i^{\lambda}(x_i) = \alpha(\lambda)\hat{U}_i(x_i) + \beta_i(\lambda) \text{ for all } i \in \mathbb{N}.$$

For $x_i = 1$, this equation implies $\hat{U}_i(\lambda) = \beta_i(\lambda)$. Therefore

$$\hat{U}_i(\lambda x_i) = \alpha(\lambda)\hat{U}_i(x_i) + \hat{U}_i(\lambda) \text{ for all } i \in \mathbb{N} \text{ and } \lambda > 0.$$

Fix any $\lambda_1, \lambda_2 > 0$. Then the above equation implies both

$$\hat{U}_i(\lambda_1 \lambda_2 x_i) = \alpha(\lambda_1 \lambda_2)\hat{U}_i(x_i) + \hat{U}_i(\lambda_1 \lambda_2)$$

and

$$\begin{aligned}
\hat{U}_i(\lambda_1 \lambda_2 x_i) &= \alpha(\lambda_1)\hat{U}_i(\lambda_2 x_i) + \hat{U}_i(\lambda_1) \\
&= \alpha(\lambda_1)\alpha(\lambda_2)\hat{U}_i(x_i) + \alpha(\lambda_1)\hat{U}_i(\lambda_2) + \hat{U}_i(\lambda_1) \\
&= \alpha(\lambda_1)\alpha(\lambda_2)\hat{U}_i(x_i) + \hat{U}_i(\lambda_1 \lambda_2).
\end{aligned}$$

Therefore $\alpha(\lambda_1 \lambda_2) = \alpha(\lambda_1)\alpha(\lambda_2)$ for every $\lambda_1, \lambda_2 > 0$. Note that $\alpha(\lambda)$ must be continuous since $\hat{U}_i$ is continuous and not a constant zero function. By Lemma 29, either $\alpha(\lambda) = 0$ or there exists $\rho \in \mathbb{R}$ such that $\alpha(\lambda) = \lambda^{\rho}$. But $\alpha(\lambda) = 0$ is not possible since $\hat{U}_i$ is strictly increasing.

Case 1: $\rho = 0$. Then for all $i \in \mathbb{N}$, we have

$$\hat{U}_i(\lambda x_i) = \hat{U}_i(x_i) + \hat{U}_i(\lambda) \text{ for all } \lambda, x_i > 0.$$

By Lemma 28, there exists $\hat{\alpha}_i$ such that $\hat{U}_i(x_i) = \hat{\alpha}_i \ln(x_i)$. Set $\hat{\beta}_i \equiv U_i(1)$. Then

$$U_i(x_i) = \hat{\alpha}_i \ln(x_i) + \hat{\beta}_i.$$

Note that $\hat{\alpha}_i > 0$ since U_i is strictly increasing.

Case 2: $\rho \neq 0$. Fix $\hat{\lambda} > 1$. Then for every $i \in \mathbb{N}$, we have

$$\hat{U}_i(\hat{\lambda} x_i) = \hat{\lambda}^\rho \hat{U}_i(x_i) + \hat{U}_i(\hat{\lambda})$$

and

$$\hat{U}_i(\hat{\lambda} x_i) = x_i^\rho \hat{U}_i(\hat{\lambda}) + \hat{U}_i(x_i).$$

Combining these equations we get

$$\hat{U}_i(x_i) = \frac{\hat{U}_i(\hat{\lambda})}{\hat{\lambda}^\rho - 1} \left(x_i^\rho - 1 \right).$$

Setting $\hat{\alpha}_i \equiv \frac{U_i(\hat{\lambda}) - U_i(1)}{\hat{\lambda}^\rho - 1}$ and $\hat{\beta}_i \equiv \frac{\hat{\lambda}^\rho U_i(1) - U_i(\hat{\lambda})}{\hat{\lambda}^\rho - 1}$, we get

$$U_i(x_i) = \hat{\alpha}_i x_i^\rho + \hat{\beta}_i.$$

Note that $\hat{\alpha}_i \rho > 0$ since U_i is strictly increasing.

Appendix F. Proof of Theorem 4

Showing that Strict Claims Monotonicity is necessary is a straightforward exercise. So suppose S satisfies the stated axioms. By Theorem 1, there exists $U \in \mathcal{U}$ such that $S = ES^U$. By way of contradiction, suppose $i \in \mathbb{N}$ satisfies $\lim_{x_i \to 0} U_i(x_i) \equiv a_i > -\infty$. Fix $j \neq i$. Choose $\epsilon > 0$ smaller than the range of U_i and U_j. I.e. set $b_i \equiv \lim_{x_i \to \infty} U_i(x_i)$, $a_j \equiv \lim_{x_j \to 0} U_j(x_j)$, and $b_j \equiv \lim_{x_j \to \infty} U_j(x_j)$, then choose $\epsilon \in (0, \min\{b_i - a_i, b_j - a_j\})$. Set $c_i \equiv U_i^{-1}(a_i + \epsilon) > 0$. Since U_i is strictly increasing we have $U_i(c_i) - U_i(x_i) < \epsilon$ for every $x_i \in (0, c_i)$. Because $\epsilon < b_j - a_j$, there exists $x_j, c_j \in \mathbb{R}_{++}$ such that $x_j < c_j$ and $U_j(c_j) - U_j(x_j) > \epsilon$. Since $S \in \mathcal{ES}$, we must have

$$S\left(\{i,j\}, (c_i, c_j), x_j\right) = (0, x_j).$$

But then Strict Claims Monotonicity implies that $S_i\left(\{i,j\}, (c_i', c_j), x_j\right) < 0$ for $c_i' \in (0, c_i)$, which is impossible.

Appendix G. Proof of Theorem 5

Showing that Equal Treatment of Equals is necessary is a straightforward exercise. So suppose S satisfies the stated axioms. By Theorem 1, we have $S \in \mathcal{ES}$. Let $U \in \mathcal{U}$ be the equal sacrifice representation of S. To show $S \in \mathcal{ES}^*$, Proposition 1 implies that it is sufficient to show that $U_i - U_j$ is constant for all i and j. But if i and j were such that $U_i - U_j$ was not constant, then one could easily construct a problem that violated Equal Treatment of Equals.

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