Queue-Rationed Equilibria with Fixed Costs of Waiting

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Abstract

The welfare impact of price controls is examined here in an exchange economy where agents may need to queue in order to make a transaction. Time spent in the queue is an endogenously-determined transaction cost, which agents take as given and which adjusts so as to clear markets when prices are prevented from performing this function. When queuing is required, it enters the household's decision as a fixed cost, rather than increasing in proportion to the amount of good exchanged, as is far more common in the previous literature. Existence of competitive equilibrium is established for this general equilibrium model. Price controls are shown to cause notable inefficiencies, which differ from those of a proportional cost model. Moreover, in certain environments, price controls will unambiguously harm all individuals relative to a Walrasian equilibrium.

1 Introduction

When a government controls prices, there is no assurance that markets will clear. At an artificially low price, buyers typically would like to purchase more of the good than sellers are willing to offer. This excess demand means that some consumers who want to purchase the controlled good will be unable to do so — that is, being willing and able to pay the posted price no longer guarantees a "right to purchase" the good. As a result, consumers will spend effort or resources in order to acquire that right.[1]

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[1]This competition for the right to purchase is aptly described in Barzel (1974) and Cheung (1974).

One of the most common methods of establishing a right to purchase is by arriving at the retail location before other customers do. Customers might arrive hours before the store opens in order to be among the first to enter and purchase the item in short supply. Thus, a queue forms, and a person's position in the queue determines his claim on the good.[2] The gasoline price controls in the United States during the 1970s provide a dramatic example of this queuing response.

A similar story can be told for suppliers facing artificially high prices. Since not all suppliers will be able to sell their desired amount, they queue up for customers. For instance, if regulated taxi fares give special advantage to some locations, such as an airport pickup, taxi drivers may form a line to wait for customers at that location. For either buyers or sellers, a sufficiently large cost of queuing will discourage participation in the market. Thus, if prices are not allowed to freely adjust, queues can take their place in equalizing supply and demand. Yet this system of rationing destroys resources in the economy, reducing time available for either leisure or labor.

Nearly all models of rationing by queues assume that the waiting cost for purchasing a good is proportional to the amount of good purchased.[3] For example, if 10 minutes of waiting are required to buy one gallon of gas, then 20 minutes must be sacrificed to buy two gallons, and so forth. The per-unit time in the queue is simply added on top of the government-dictated dollar price. The main reason for making such an assumption is undoubtedly for technical properties: as a proportional cost, the queue resembles an increase in the price, much like a tax wedge, and many of the standard tools of analysis can be used.

However, it is difficult to reconcile such a model with a typical story of customers waiting in line to make their purchases, as most authors have acknowledged. A number of situations come to mind in which consumers must "wait their turn" in a queue, but having reached the head of the queue, are free to purchase as much as their budget allows. When considering gas shortages (if quantity rations are not imposed), the length of waiting time would be the same whether the consumer purchases 5 or 16 gallons once at the pump. Indeed, the owner of a motor-home or Hummer could potentially purchase 30 or more gallons. Another instance would be a person waiting for a taxi, who can potentially purchase as much distance

[2]There are, of course, other methods by which consumers might try to acquire this right to purchase, such as giving some form of kickback to the seller, or lobbying for the government to assign the rights, perhaps through rationing coupons or lotteries. These would be more likely to take place in the face of long-standing shortages. However, in the initial period after a regime of government-dictated prices is imposed, queuing is much more likely to arise. Products are normally sold on a first-come, first-served basis; hence, anticipating a shortage, consumers can immediately begin using this technique to make their claim. Bribes or lobbying would only evolve as participants in the market gradually find sellers willing to circumvent the law or legislators willing to rewrite the law.

[3]Examples include Kornai and Weibull (1978), Bucovetsky (1984), Stahl and Aleexev (1985), Nguyen and Whalley (1986), Sah (1987), Stahl (1987), Nguyen and Whalley (1990), Boycko (1992), Osband (1992), and Polterovich (1993).

as affordable once he obtains a cab.

In many markets, the fixed cost of waiting seems more appropriate for observed behavior,[4] yet the literature using this assumption is remarkably sparse. Weitzman (1991) argues for this fixed cost specification, which he used in a partial equilibrium model depicting shortages due to hoarding. He assumes that the time spent in the queue each shopping trip does not vary with the amount purchased each trip (though agents can choose how frequently to make shopping trips). Stahl (1985) presents the only other model in which queue costs do not vary with quantity purchased. His primary interest, as in his proportional queue cost model (Stahl, 1987), was to determine whether a queue-rationed equilibrium would be stable in the sense of tâtonnement theory, and thus did not investigate welfare issues that are the central interest of this work.

This paper presents a general equilibrium model of an exchange economy under a regime of price controls. Queues are also allowed to form in each market, which adjust to restore equilibrium when prices are restricted from doing so. Households can be heterogeneous in both preferences and endowments, enabling a consideration of distributional consequences of price controls. In particular, preferences may differ with respect to leisure time sacrificed in waiting, which can affect the composition of who purchases the price-controlled goods.

To allow appropriate comparison of welfare outcomes in an otherwise identical environment, the model is specified to accommodate either fixed costs or proportional costs of waiting.[5] Under the proportional cost assumption, the model resembles those provided by Nguyen and Whalley (1986)[6] and Stahl (1987). Under the fixed cost assumption, the model is similar to Stahl (1985). However, the significant difference is that nearly all models in the literature (including the three just mentioned) have a government that either decrees all prices in the economy, or that controls only one price. The model presented here is more general, allowing the government to specify a range of acceptable prices for each market, or even leave some markets unregulated.

[4]The assumption of proportional cost would be most applicable if the retailer or government has imposed a "one per transaction" policy; thus, buying multiple units would require several trips through the queue, and the time spent passing through the queue once would be the per-unit cost. To be plausible, agents would need sufficient time to make these trips multiple times in the same period. Even then, a technical problem is that goods are perfectly divisible in these models; hence, if a fraction of a unit is purchased, the consumer only waits that same fraction of the queue time.

[5]These two specifications could be combined into a general, non-linear cost of waiting; however, doing so adds very little to the results. The fixed-cost-only and proportional-cost-only equilibria would both still exist, as well as a continuum of equilibria between; yet the model would offer no selection criteria to predict which equilibrium would occur. Moreover, the main purpose of this paper is to examine whether the more natural set up of fixed costs of waiting significantly differs in welfare results from the common proportional cost model, and this is more effectively accomplished by examining the extreme cases.

[6]Nguyen and Whalley refer to "endogenous transaction costs," rather than specifically considering queues; however, these costs are proportional to the amount traded and, like queues, result in the destruction of real resources.

Treating queues as a fixed cost significantly alters equilibrium behavior and the efficiency of a queue rationing system. While both scenarios result in wasting some of the aggregate resources, proportional cost queues introduce marginal distortions, similar to a per-unit tax. Fixed cost queues act somewhat like a lump sum tax, and hence the inefficiency is limited to the lost endowment; however, additional distortion can arise when agents are discouraged from participating in a particular market due to the high fixed cost of entering it. The lack of marginal distortions reduces the incentive for black market trading after equilibrium is reached.

Even so, queue times tend to be longer under *fixed cost* queues, since the income effect is more of a blunt tool in reducing demand for a particular good, compared to the price effect of a proportional cost queue. As a result, it is often the case that proportional cost models are overly optimistic as to the welfare consequences of rationing by queues,[7] and that all agents are actually worse off under price controls than they would be under market clearing prices.

The proof of the theorems for existence of equilibrium require some technical sophistication, which partly explains why fixed cost queues have been neglected in the literature. Even so, the existence results are quite positive — minimal restrictions are needed on preferences, endowments, or price regimes. The technical challenge in this model is that the fixed cost of queuing introduces non-convexities into the consumer's budget set. This poses similar difficulties to those created by non-convex preferences — namely, the demand correspondence (here, a function of both queue times and prices) could be non-convex valued. This is overcome by having a continuum of each type of agent. Agents of the same type may choose different, equally-preferred bundles; by assigning the right fraction of them to various bundles, any convex combination can effectively be obtained, as in Aumann (1966).

Controlling prices at non-Walrasian levels could certainly impact production decisions; however, in order to focus on individual decisions to queue, most research in this area has abstracted from production issues by either considering exchange economies or conducting partial equilibrium analysis.[8] Here I pursue the first option. The agents in the economy are each endowed with a bundle of goods; each market will have an associated queue time which represents the leisure that either buyers or sellers[9] of that good must sacrifice if they wish to trade that good.

[7] Sah (1987), Osband (1992), and particularly Polterovich (1993) compared welfare under proportional cost queues to welfare under other rationing methods, such as free markets, coupons, lotteries, or black markets. The typical result is that markets are favored by the wealthy, while the poor favor queues or coupons with resale. No previous work has compared proportional cost queues to fixed cost queues.

[8] Some exceptions are Bucovetsky (1984), Nguyen and Whalley (1990), and Osband (1992), where production is incorporated. The unfortunate consequence of excluding production is that this model is not applicable to wage controls. Even so, queuing is not typically used as a rationing method in the labor market.

[9] The model is specified such that it is impossible to simultaneously have a queue of buyers and sellers. The transaction itself takes no time, and hence, the shorter of the two queues would immediately clear.

Queue rationing differs from the quantity rationing literature on price rigidities, such as the exchange economy modeled in Drèze (1975). There, equilibrium is restored in spite of price controls through the use of endogenously-determined restrictions on quantities traded. No institutional details are included in the model, such as government ration coupons, and it is hard to tell a convincing story of how those quantity restrictions would endogenously adjust, unlike the rather intuitive (and decentralized) concept of people increasing their wait time if a good is currently in short supply.

In discussing this topic, the use of the term "queue" might suggest that I will make use of "queuing theory," but this is not the case. In the model presented here, the transactions themselves are assumed to take negligible time; hence, there is no need to model a server which is capable of processing a transaction at a certain rate (such as the time it takes to dispense gas from a pump to the vehicle's tank).

Indeed, agents in my model do not literally choose an arrival time to determine their position in a queue. Rather, they are informed as to the length of time they must wait in order to gain access to the market, and then decide whether it is worth the wait. They take queue times as given (abstracting from the queue process that generates those times), in the same sense that prices are taken as given in a standard model (abstracting from an auction process). The agent assumes that if he spends the required amount of time in the queue, he will be able to purchase (or sell) as much as he desires — and in equilibrium, the required amount will adjust so as to make this assumption true. One might refer to this as a "perfectly competitive queue."

This approach is used in nearly all of the literature on rationing by queues (albeit with a waiting cost that is proportional to the quantity purchased). Several papers provide micro foundations of queuing decisions. Kornai and Weibull (1978) models the waiting decision in terms of flows into and out of the queue (on entering the market and on obtaining a single unit of the good, respectively). In the steady state, all agents wait the same amount of time while passing through the queue. Holt and Sherman (1982) explicitly model a waiting-line auction in the case of a single indivisible commodity with unit demand. Agents strategically select an arrival time, taking the strategies of other agents as given, thus forming a literal queue to determine which agents obtain a unit of the commodity. Platt (2007) extends this to a divisible commodity with continuous demand. As the number of agents increases, equilibrium expected utility approximates the competitive results depicted here.

In light of these results, one can appropriately interpret waiting costs here as the steady state of a queuing process. Admittedly, this will not capture the dynamic transition after policy changes or other shocks. For instance, a person who is already standing in line when a price ceiling is lowered will benefit more than those who enter the market after the queue reaches steady state. Even so, one would expect the transition to be rather brief.

The remainder of the paper proceeds as follows: Section 2 presents the two versions of the model and defines equilibrium for rationing by queues. Section 3 then establishes existence of equilibrium under rather minimal restrictions. Next, the inefficiencies caused by proportional and fixed cost queues are contrasted in Section 4, which is developed further in Section 5 by examining cases in which all agents are strictly worse off under price controls. Section 6 offers conclusions and direction for further work.

2 Model

Consider an exchange economy, with N types of agents and a set of goods $\{0, 1, \ldots, \ell\}$, where good 0 is the numeraire good. Assume that each agent type has a unit measure of agents. Each agent i has a consumption set of $X_i = \mathbb{R}_+^{\ell+1}$, and an endowment $\bar{\omega}_i = (\omega_{i0}, \omega_i) \in X_i$. We assume $\omega_{i0} > 0$ for all i, and $\sum_{i=1}^N \omega_i \gg 0$. For any allocation or price vector, we use the bar notation $\bar{x}_i = (x_{i0}, x_i)$ for identifying the combination of the numeraire with other goods.

Agents also have preferences $\succcurlyeq_i$ defined on X_i, which are assumed to be continuous, convex, strictly monotone, complete preorders.

All prices are normalized in terms of good 0 (which is permissible since preferences are strictly monotone), and are expressed as a vector $p \in \mathbb{R}_+^\ell$. Similarly, price controls (when imposed) are also specified relative to this normalization. The government sets a permissible range for the price of each non-numeraire good $j \in \{1, \ldots, \ell\}$ between a price floor, p_j^F, and a price ceiling, p_j^C, requiring that $p_j^F \leq p_j \leq p_j^C$. The government can effectively decree a particular price for good j by setting $p_j^F = p_j^C$. On the other extreme, either side of the controls can be "turned off" by setting $p_j^F = 0$ or $p_j^C = +\infty$. Note that although the market for good 0 is never controlled directly, the government can effectively control the relative price between it and any other good.

The numeraire plays one other important role: any queue costs must be paid in units of the numeraire. If one interprets good 0 as leisure, then the queue cost can literally be thought of as time. But more generally, one could also interpret it as any other valuable commodity which is sacrificed in order to secure a right to purchase the good.[10] The key assumption is that this numeraire is destroyed when used to queue, rather than being transfered it to the seller (as in a bribe).

The market for each non-numeraire good $j \in \{1, \ldots, \ell\}$ has an associated queue time, $t_j \in \mathbb{R}$. In the proportional cost model, this represents the *per-unit cost* of purchasing the good, which can be thought of as time spent waiting in line. The "proportional" nature of this waiting time is quite literal — since the goods are perfectly divisible, an agent can buy

[10] Those who camp on store sidewalks for days to obtain a scarce product certainly sacrifice comfort and perhaps health!

a fraction of good j and wait only an equal fraction of t_j. If $t_j > 0$, anyone who wishes to purchase good j must sacrifice t_j units of time per unit of j purchased. If instead $t_j < 0$, would-be sellers must give up $-t_j$ units of time per unit sold.

In the fixed cost model, t_j is similar to a membership fee for access to the market. If $t_j > 0$, anyone who wishes to purchase any amount of good j must sacrifice t_j units of time. If instead $t_j < 0$, would-be sellers must give up $-t_j$ units of time to be able to participate in the market. Unlike a two-part tariff, however, this membership fee is not received by the other party to the transaction; rather, this portion of the agent's endowment is destroyed. The waiting cost remains the same regardless of the quantity purchased (or sold). Moreover, so long as they sacrifice t_j, agents expect to be able to complete their intended purchase (or sale), which will be true in equilibrium.

Thus, for a given $t \in \mathbb{R}^\ell$, if an agent makes exchanges of the non-numeraire goods $y \in \mathbb{R}^\ell$ (where positive values indicate acquired goods and negative for sold goods), he must wait in line for time $Q^o(y, t)$, defined for proportional costs as:

$$Q^\rho(y,t) \equiv \sum_{j=1}^{\ell} \max\{0, t_j \cdot y_j\} \tag{1}$$

or in the case of fixed costs:

$$Q^f(y,t) \equiv \sum_{j=1}^{\ell} (\max\{0, t_j \cdot I(y_j > 0)\} + \max\{0, -t_j \cdot I(y_j < 0)\}) \tag{2}$$

where $I(\cdot)$ is the indicator function.

Prices and queue times are both taken as given by agents. For either fixed or proportional cost queues, the constraint set for agents of type i is thus:

$$\begin{aligned} B^o(p,t;\bar{\omega}_i) \equiv \{(x_0, x) \in X_i : \; & \exists \bar{y} \in \mathbb{R}^{\ell+1} \text{ s.t. } x \le \omega_i + y, \\ & x_0 \le \omega_{i0} + y_0 - Q^o(y,t), \\ & \text{and } y_0 + p \cdot y \le 0\} \end{aligned} \tag{3}$$

The vector of net trades, y, is included so as to determine how much an agent must queue, if at all. Note that there is a market for good 0: the agents can purchase or sell the numeraire, as well as using the numeraire to queue.[11]

[11]For instance, an agent could sell his leisure time to others, perhaps to wait on their behalf. However, if he sells that time, he cannot use it for his own queuing — *i.e.* he cannot simultaneously wait for himself and for others. The model can be amended to allow one person to wait in line and purchase for multiple people, but it would not significantly change the results. If all agents were to "team up" in pairs, with only one waiting at a time, the cost of waiting will be cut in half. This will increase demand for that good; to restore equilibrium, the wait time must rise until it is twice its original length. Barzel (1974) makes this argument. We also assume no black markets operate, where goods can be sold at prices forbidden by the government; this could have a significant impact on results and is an interesting topic to pursue in future work.

Moreover, agents are allowed to freely dispose of some of their endowment rather than queue to sell it (even though they would never choose to do so in equilibrium, due to strictly monotone preferences). For $t = 0$, $B^o(p, t; \bar{\omega}_i)$ becomes the typical budget set. When $t \neq 0$, $B^\rho(p, t; \bar{\omega}_i)$ will be kinked but will still be convex, while $B^f(p, t; \bar{\omega}_i)$ will not be convex. The following lemma provides some basic properties of this constraint set; the proof is straightforward, and is presented in Appendix **??**.

Lemma 1. *For any $p \gg 0$ and any t, $B^o(p, t; \bar{\omega}_i)$ is non-empty, closed-valued, and continuous at (p, t). Moreover, $B^\rho(p, t; \bar{\omega}_i)$ is convex-valued.*

Agents pick the most preferred bundle from within this set. Hence, the demand correspondence is defined as:

$$\bar{\xi}_i^o(p, t; \bar{\omega}_i) \equiv \{\bar{x} \in B^o(p, t; \bar{\omega}_i) : \ \bar{x} \succsim_i \hat{x}, \forall \hat{x} \in B^o(p, t; \bar{\omega}_i)\} \tag{4}$$

For notational convenience, we typically omit $\bar{\omega}_i$ from the listed parameters of $\bar{\xi}_i^o$. As an almost immediate result of the properties of the budget set and the assumptions on preferences, demand has the following properties:

Lemma 2. *For any $p \gg 0$:*

1. *$\bar{\xi}_i^o(\cdot\,,\cdot\,)$ is non-empty, compact-valued, and upper hemicontinuous*
2. *$\bar{\xi}_i^\rho(\cdot\,,\cdot\,)$ is convex-valued*
3. *$x_{i0} - \omega_{i0} + Q^o(x_i - \omega_i, t) + p \cdot (x_i - \omega_i) = 0 \quad \forall \bar{x}_i \in \bar{\xi}_i^o(p, t)$ (Walras' Law).*

Due to the non-convexity of the budget $B^f(p, t)$, we could have some agents whose demand correspondence $\bar{\xi}_i^f(p, t)$ is not convex-valued. Even so, demand is well-defined for all $t \in \mathbb{R}^\ell$ and all $p \gg 0$. If the demand correspondence is multi-valued at any particular (p, t), we can obtain an average allocation from any convex combination of the allocations in $\bar{\xi}_i^f(p, t)$ by assigning the right fraction of agents of that type to the various bundles that are in $\bar{\xi}_i^f(p, t)$. Thus, we define aggregate demand as the sum of the convex hulls of the demand of each type: $\bar{\xi}^f(p, t) = \sum_{i=1}^N co(\bar{\xi}_i^f(p, t))$.

If demand is multi-valued under $\bar{\xi}_i^\rho(p, t)$ for a particular p and t, its members must form a convex set. Thus, even if agents of a given type choose different allocations, we may proceed as though they all choose the same allocation which is an average of the different bundles. We define aggregate demand as the sum of the demand of each type: $\bar{\xi}^\rho(p, t) = \sum_{i=1}^N \bar{\xi}_i^\rho(p, t)$.

Markets clear when, for some (p, t) and some $(x_0, x) \in \bar{\xi}^o(p, t)$, $x = \sum_{i=1}^N \omega_i$. (Due to Walras' Law expressed in Lemma 2, the numeraire market must clear if all the others do.

This is convenient, since it spares us from needing to track the aggregate queuing time.) Define aggregate excess demand correspondence to be $\bar{\eta}(p,t) = \bar{\xi}(p,t) - \sum_{i=1}^{N} \bar{\omega}_i$, though we will typically be interested in just the last ℓ terms, $\eta(p,t)$.

In this model, a *Walrasian Equilibrium (WE)* is an allocation $(\bar{x}_i^*)_{i=1}^N$ and prices p^* such that:

1. $p_j^F = 0$ and $p_j^C = +\infty$ for all j.
2. $\bar{x}_i^* \in \bar{\xi}_i^o(p^*, 0)$ for all i.
3. $\sum_{i=1}^N x_i^* = \sum_{i=1}^N \omega_i$.

We say that a particular allocation $(\bar{x}_i)_{i=1}^N$ can be *supported* as a Walrasian equilibrium (with redistribution) if there exists a p such that $\bar{x}_i \in \bar{\xi}_i(p, 0;\, \bar{x}_i)$ for all i and $\sum_{i=1}^N \bar{x}_i \leq \sum_{i=1}^N \bar{\omega}_i$.

When prices are unrestricted and no queues occur, this coincides with the typical Walrasian equilibrium in an exchange economy. The standard existence theorem and the first and second welfare theorems would apply to the WE.

With controls in place, queuing times will have to adjust so as to clear the market by discouraging either buyers or sellers from participating (either on the intensive margin, reducing the quantity traded by each individual, or on the extensive margin, by reducing the number of agents who enter the market). Given price controls p^F and p^C, a *Proportional Cost Queue-Rationed Equilibrium (ρ-QRE)* is an allocation $(\bar{x}_i^*)_{i=1}^N$, prices p^*, and queue times t^* such that:

1. $p^F \leq p^* \leq p^C$
2. $\bar{x}_i^* \in \bar{\xi}_i^\rho(p^*, t^*)$ for all i
3. $\sum_{i=1}^N x_i^* = \sum_{i=1}^N \omega_i$
4. For all j, if $t_j^* > 0$ then $p_j^* = p_j^C$, and if $t_j^* < 0$ then $p_j^* = p_j^F$.

while a *Fixed Cost Queue-Rationed Equilibrium (f-QRE)* would replace the second condition with $\bar{x}_i^* \in co(\bar{\xi}_i^f(p^*, t^*))$ for all i. While each agent faces the same queue time t_j^* in market j, the heterogeneous preferences allow households to value that lost numeraire differently.

These definitions of equilibrium coincides with those in Stahl (1985, 1987), except for the addition of the fourth condition. This states that waiting only occurs in markets where the price has reached a bound. In Stahl's work, all prices were set by decree ($p_j^C = p_j^F$ in this model), so this condition was trivially satisfied.

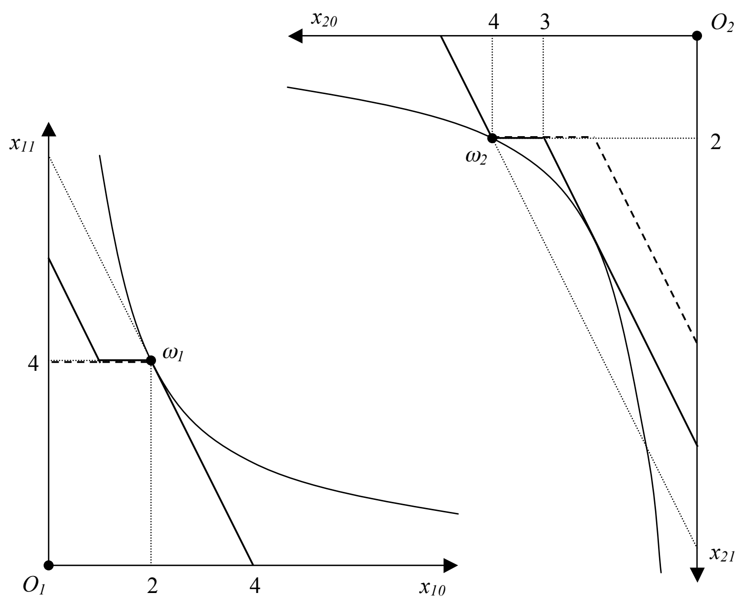

Figure 1: Budget set with fixed cost queues of $t^* = 1$ (solid) and $t^* = 2$ (dashed).

With a range of permissible prices, one must add the fourth condition to ensure that excess demand in price-controlled market j is not reduced by using a queue in some unregulated market j'. Such a situation would not fit the standard notion of equilibrium: a seller who observed queues in his unregulated market would use the opportunity to raise prices.[12] Thus, the economy would not remain at those prices very long. Thus we require that in equilibrium, buyers queue only if the price ceiling in that market has been reached, and likewise for sellers with the price floor.

Another issue with the consistency of this definition is that some QREs may require t^* to be large enough so as to fully shut down trade, thus maintaining the endowment as the equilibrium allocation. For example, consider an economy (illustrated in Figure 1) with 2 goods and 2 types of agents, where $u_1(x_{10}, x_{11}) = x_{10} \cdot x_{11}$ and $u_2(x_{20}, x_{21}) = x_{20} \cdot x_{21}$. Their endowments are $\omega_1 = (2, 4)$ and $\omega_2 = (4, 2)$. Suppose the government sets $p^F = p^C = 0.5$. Type 1 agents will choose to consume their endowment regardless of t; thus, type 2 agents must be discouraged from purchasing anything besides their endowment. In the ρ-*QRE*, this will occur as long as $t \geq 1.5$. For the *f-QRE*, any $t^* \geq 1$ will achieve this.

One may be justifiably concerned about such an equilibrium, where agents expect a positive queue time for a market in which no agent would rationally choose to participate. In the ρ-*QRE* example above, if $t^* \geq 1.5$, none of the agents in the model will buy or sell

[12]Unlike standard competition, he could raise his price (in small amounts) without losing all of his customers, since some customers would value the shorter wait over the price increase.

good 2. If so, why should a potential buyer expect a per-unit wait of t^* if he instead tried participating in the market? This feature can occur in any queue rationing model — indeed, if prices are held in a position that drives out all participants on one side of the market, the only possibility for queues to "restore equilibrium" is to drive away the other side of the market with extremely long waits. This unpleasant feature is rarely identified in the many queue rationing models in which it occurs.[13]

The fixed cost model offers a refinement that identifies a "reasonable" subset of QRE, weeding out the less plausible equilibria. We define an *Essential Queue-Rationed Equilibrium* as an *f-QRE* $\left((\bar{x}_i^*)_{i=1}^N, p^*, t^*\right)$ such that for each good $j \in \{1, \ldots, \ell\}$, either $t_j^* = 0$ or there exists some i and some $\bar{x}_i \in \bar{\xi}_i(p^*, t^*)$ such that $t_j \cdot (x_{ij} - \omega_{ij}) > 0$. Note that $\bar{x}_i$ need not be the same as $\bar{x}_i^*$. This indicates that, in an Essential QRE, agents only expect to find a queue if some agents are willing to participate in that market (even if only measure 0 of agents actually do).

Thus, in the previous example, the equilibria with $t^* > 1$ would not be rational, since no agent weakly prefers participating in the market. When $t^* = 1$, however, agents of type 2 are indifferent about participating in the market, and could reasonably expect that waiting 1 unit of time would be necessary if someone were to enter the market. In the next section, it is shown that we can restrict ourselves to Essential QRE in the fixed cost model without loss of generality.

3 Existence of QRE

In the fixed and proportional cost models presented in Stahl (1985, 1987, respectively) the government decrees a single vector of prices, and queues must adjust to restore equilibrium. Thus, one cannot simply obtain existence of QRE from the proofs of those papers if the government instead sets a range of acceptable prices in each market.[14] Both prices and queues may have to adjust, and in particular, queues can only be used once prices have been exhausted. In typical fashion, the existence argument for Queue-Rationed Equilibria depends on finding the fixed point of a well-chosen correspondence. Choosing the appropriate space for prices and queues is the most critical element of the proof.

For proportional costs, "official prices" are represented on a (constrained) simplex, while queue times are translated into an effective "total price" per unit, which can also be repre-

[13] This can be considered an artifact of the model, caused by not explicitly modeling the queue. If we were literal about the queue, a buyer entering such a market would have to wait infinitely before a seller would arrive.

[14] If one dispenses with the fourth condition for a QRE, then the equilibrium concept coincides with Stahl's. In that case, one can show that for each admissible price vector p, some t^* exists which forms an equilibrium. To prove this requires some modification of the proof by Stahl, since he assumed differentiability; but this is not a necessary condition for existence, and working without it is not particularly difficult.

sented on the unit simplex. In responding to excess demand, the total price for a good is only allowed to deviate from the official price if that market is close to its ceiling or floor. In the limit, these deviations (which represent non-zero queues) only occur if the price control is binding. The proof is provided in Appendix **??**.

Theorem 1. *For any given price control policy, p^F and p^C, where $p_j^F < +\infty$ and $p_j^C > 0$ for all j, there exists a ρ-QRE.*

Existence in the fixed cost model is significantly more challenging to establish, since queue times cannot be confined to a simplex. The concept of a per-unit "total price" would be inappropriate, since the price of waiting is non-linear in the quantity purchased. Rather, queues are bounded in absolute value by the length of time that would be entirely unaffordable (even for the wealthiest agents when they use all of their income to purchase time from others). As in the previous proof, queues are forced to zero unless the price in that market is near its ceiling or floor. (This proof also appears in Appendix **??**.)

Theorem 2. *For any given price control policy, p^F and p^C, where $p_j^F < +\infty$ and $p_j^C > 0$ for all j, there exists a f-QRE.*

The only restriction on existence is to exclude cases when the price of some good is forced to be zero, or when it is forced to be infinite, effectively causing the price of the numeraire to be zero. In either case, no finite fixed cost queue would be sufficient to discourage agents from waiting the required amount, then demanding an infinite amount of the good with zero price.

The preceding theorem does not prevent queues from being so long that markets are entirely shut down — and in fact, they may need to be. However, the following theorem demonstrates that any *f-QRE* can also be supported as an Essential QRE, using the same allocation and prices, but reducing queue times to the point that some agents would at least be willing to participate in each market. Thus, we lose nothing by requiring the agents to have rational expectations of the queue times.

Theorem 3. *For any f-QRE $\left((\bar{x}_i^*)_{i=1}^N, p^*, t^*\right)$, the allocation $(\bar{x}_i^*)_{i=1}^N$ can be supported as an Essential QRE using the same prices and the either the same or shorter queue lengths.*

The proof is provided in Appendix **??**. Combined, Theorems 2 and 3 imply that, under fixed costs of waiting, an Essential QRE exists for every price control policy, with queues only occurring in markets where the price control is binding.

We have no assurance as to the uniqueness of equilibrium in either model. Even a particular vector of prices can have multiple QREs associated with it. This model provides no method of selection among multiple equilibria. One interesting case occurs when all agents have quasi-linear utility with respect to the numeraire. If the Walrasian equilibrium of the economy is not at a corner solution, then multiple *f-QRE* exist for the *same* prices that support the Walrasian equilibrium. Of course, one QRE will be the Walrasian equilibrium with no queues; yet others will occur with non-zero queues (though small enough so that no one is discouraged from participating in the market).[15]

Thus, in such an environment, even if the government found the prices that support a Walrasian equilibrium (perhaps by using a computational general equilibrium model) and decreed these to be the only permissible price, queues may still form. This could be interpreted as agents having self-fulfilling expectations that queues will form. Since they strictly benefit from trade, agents are willing to sacrifice some time to still have access to that market. This might be particularly applicable to command economies where agents have grown accustomed to rationing by queues. If the government happens to choose the "right" price, there is no guarantee that queues will disappear.

4 QRE and Welfare Theorems

One question of interest is whether queue-rationed equilibria are Pareto efficient. If we judge using the traditional definition of Pareto efficiency, the answer would be: not if any queuing is needed to reach the equilibrium outcome. This is because goods that are valued are destroyed in the process of queuing, and given the strictly monotone preferences, the lost good could have been redistributed among all agents for a strictly preferred allocation.

Define the set of *Pareto efficient allocations* as the set:

$$\mathbb{P}(\bar{\omega}) \equiv \Bigg\{ \bar{x} \in \mathbb{R}^{(\ell+1)\cdot N} : \sum_{i=1}^{N} \bar{x}_i - \bar{\omega}_i \leq 0 \text{ and } \forall \bar{x}^a \text{ where } \sum_{i=1}^{N} \bar{x}^a_i - \bar{\omega}_i \leq 0, \\ \text{either } \exists i \text{ s.t. } \bar{x}_i \succ_i \bar{x}^a_i, \text{ or } \forall i\ \bar{x}_i \not\succ_i \bar{x}^a_i \Bigg\} \tag{5}$$

Proposition 1. *If $(\bar{x}^*_i)_{i=1}^N$, p^*, and t^* constitute a (ρ- or f-)QRE and there is no p' that can support $(\bar{x}^*_i)_{i=1}^N$ as a WE, then $\bar{x}^* \notin \mathbb{P}(\bar{\omega})$. In particular, if for some i and some j, $t^*_j \neq 0$ and $(x^*_{ij} - \omega_{ij}) \neq 0$ then $\bar{x}^* \notin \mathbb{P}(\bar{\omega})$.*

The first statement identifies all Pareto inefficient QRE. The second statement describes an important subset of inefficient QRE — namely, those in which anyone spends time in a queue. The difference between these two sets would be equilibria in which a queue shuts down a market that would otherwise permit mutually beneficial trade.

[15]This phenomenon is also possible in a *ρ-QRE*, but it is difficult to distinguish a class of preferences that consistently produce the result. For fixed cost queues, the claim has been proven by the author for all quasi-linear preferences.

Proof. The first and second welfare theorems apply to WE in this model; hence, an allocation can be supported as a WE if and only if it is Pareto optimal. This verifies the first statement in the proposition.

In particular, if anyone waits in queues, we can identify the Pareto improvement: By market clearing, we know that $\sum_{i=1}^{N} x_i^* - \omega_i \leq 0$. Let $T > 0$ be the aggregate time spent in queues; *i.e.* $T \equiv \sum_{i=1}^{N} \sum_{j=1}^{\ell} \max\{0,\ t^*\ (x_{ij}^* - \omega_{ij})\}$ for a ρ-*QRE*, and $T \equiv \sum_{i=1}^{N} \sum_{j=1}^{\ell} (\max\{0, t_j \cdot I(y_j > 0)\} + \max\{0, -t_j \cdot I(y_j < 0)\})$ for a f-*QRE*. In either case, $\left(\sum_{i=1}^{N} \omega_{i0}\right) - T \geq \sum_{i=1}^{N} x_{i0}^*$

Let $x_{ij}^a = x_{ij}^*$ for all i and all $j \in \{1, \ldots, \ell\}$, and let $x_{i0}^a = x_{i0}^* + (1/n) * T$ for all i. By strict monotonicity of preferences, $\bar{x}_i^a \succ_i \bar{x}_i\ \forall i$; yet $\sum_{i=1}^{N} \bar{x}_i^a - \bar{\omega}_i \leq 0$. □

While this illustrates the clear waste involved in rationing by waiting, it is unfair to compare a controlled market that incurs transaction costs to a social planner who can redistribute goods without burning up the numeraire. Instead, let us ask if the resulting allocation can be improved upon, given that some of the numeraire is already lost.

Several interpretations can be given to this question. One might assume that a social planner faces the same total cost of redistribution in the process of reallocating goods. For instance, if agents can queue up to receive the redistribution, they would likely compete until a similar amount of time is lost as in the QRE. We then pose the question whether the planner can do any better, given that the same amount of the aggregate endowment will be destroyed. This resembles the analysis of efficiency in taxation in public finance literature.

In that vein, Theorem 4 establishes that proportional cost queues in general are not efficient, even in this sense of constrained efficiency. The queues introduce marginal distortions, placing wedges between the effective price faced by buyers versus sellers in the various markets. On the other hand, Theorem 5 demonstrates that fixed cost queues are efficient, given that the lost queue time is unavoidable. Since the queue effectively reduces the wealth of those who participate, it loosely resembles a lump sum tax. The exception is when the queue discourages some otherwise willing buyers or sellers from participating in the market; in such a case, a social planner may still be able to make Pareto-improving redistributions on the f-*QRE* allocation.

A different interpretation of these results is to consider the incentives that agents face after the economy attains a queue-rationed equilibrium and agents have received their allocations. At that point, time previously spent in the queue is sunk and no longer part of the aggregate endowment. We then might ask if any mutually beneficial trades could be arranged, even at non-official prices. In the case of a ρ-*QRE* with non-zero queues, there is always a residual incentive to trade, which makes the formation of a black market more likely. In a f-*QRE* in which no one has been discouraged from participating, however, there would be no reason to arrange for further trading after official markets have closed.

Define the set of *constrained feasible allocations* as the set:

$$\mathbb{F}_C(\bar{x}^*) \equiv \left\{ \bar{x} \in \mathbb{R}_+^{(\ell+1)\cdot N} : \sum_{i=1}^{N} \bar{x}_i - \bar{x}_i^* \leq 0 \right\} \tag{6}$$

For a particular allocation $\bar{x}^*$, $\mathbb{F}_C(\bar{x}^*)$ indicates all allocations that could be reached through the redistribution of resources in $\bar{x}^*$. Note that $\mathbb{F}_C(\bar{\omega})$ would be the typical set of all feasible allocations. In practice, we will choose $\bar{x}^*$ to be a QRE allocation, which will have the same aggregate resources as in the endowment minus the time lost via queuing.

Define the set of *constrained Pareto efficient allocations* (using the resources available in $\bar{x}^*$) to be all allocations such that any other constrained feasible allocation will either be strictly worse for at least one agent, or weakly worse for everyone. In particular:

$$\mathbb{P}_C(\bar{x}^*) \equiv \left\{ \bar{x} \in \mathbb{F}_C(\bar{x}^*) : \; \forall \bar{x}^a \in \mathbb{F}_C(\bar{x}^*), \text{ either } \exists i \text{ s.t. } \bar{x}_i \succ_i \bar{x}_i^a, \text{ or } \forall i \; \bar{x}_i \succcurlyeq_i \bar{x}_i^a \right\} \tag{7}$$

This same approach is used in Stahl and Aleexev (1985), which they refer to as *queue efficient*.[16]

Theorem 4. *Let $(\bar{x}_i^*)_{i=1}^N$, p^*, and t^* constitute a ρ-QRE. Only in the following situations is it possible for $\bar{x}^* \in \mathbb{P}_C(\bar{x}^*)$:*

- *There exists a p' that can support $\bar{x}^*$ as a WE, or*
- *$t_j \neq 0$ and every buyer or every seller of j has kinked indifference curve at $\bar{x}^*$, or*
- *$t_j \neq 0$ and every buyer i of j has hit a corner solution, where $x_{i0}^* = 0$, or similarly if every seller i of j has hit a corner solution where $x_{ij}^* = 0$.*

The first condition alone is sufficient to ensure (unconstrained) efficiency. However, for this to occur, it must be that no trade occurs in markets with non-zero queues. Assuming that trade does occurs in the WE, this condition would not occur.

The other two conditions are not sufficient to obtain constrained efficiency, but are the only other cases in which it can be achieved. Of course, one could exclude these possibilities by assuming smooth preferences with marginal utility approaching infinity as consumption

[16] In the same spirit, Drèze and Müller (1980) distinguish between unconstrained and constrained Pareto efficiency in a model of rationing by tradable coupons. The former is the traditional definition of efficiency, in which an allocation is compared to all other feasible allocation. In the latter, an allocation is compared only to other allocations where each individual's net trades have zero value at the specified price. In other words, individuals still respect their own budget, but restrictions due to coupons are ignored.

of the good drops to zero. Similarly, even allowing kinks but with strictly convex preferences, one could generically exclude the second condition.

Besides these three possible exceptions, any other situation is certain to be inefficient in this constrained sense. This is a consequence of the marginal wedge of queue times, which cause buyers and sellers to face different prices. Thus, one can be confident that in almost all situations, binding price controls and proportional cost queues will be inefficient even if the lost time is treated as sunk. The proof is provided in Appendix **??**.

Turning to fixed cost queues, the results are more positive. In many cases, fixed cost queues will cause no inefficiencies beyond the wasted time itself. The possible exception is when someone has been discouraged from participation in a market due to the queue in that market. This is not to say that everyone must participate, but rather that those who do not participate still would not even if there were no queue.

Theorem 5. *Let* $(\bar{x}_i^*)_{i=1}^N$*,* p^**, and* t^* *constitute a f-QRE. Suppose that for all* j *and all* i*, if* $\bar{x}_i \in \bar{\xi}_i(p^*, t^*)$ *where* $x_{ij} = \omega_{ij}$*, then* $\bar{x}_i \in \bar{\xi}_i(p^*, (t_1^*, \ldots, t_{j-1}^*, 0, t_{j+1}^*, \ldots, t_\ell^*))$*. Then* $\bar{x} \in \mathbb{P}_C(\bar{\omega}, t^*)$*.*

In a particular extension of this result, if *all* agents participate in the various markets in spite of queues, then we can simply reduce each buyer's (or seller's, depending on the sign) time endowment by t^* units of time and then allow the market to operate as it would in a standard exchange economy. This means that after a queue-rationed equilibrium is attained and agent receive their allocations, there would be no opportunities for further mutually beneficial trade, even at uncontrolled prices.

This somewhat surprising result will not hold if some agents were discouraged from participation in a market due to the equilibrium queues t^*. Those agents will still have a residual incentive to make trades if a black market were to open.

5 Pareto Dominance

With the traditional definition, two Pareto efficient allocations cannot be compared except by aggregating with a social welfare function. This is not always the case with two *constrained* Pareto efficient allocations, because each allocation may have a different level of total resources available (*i.e.* one may have required more queue time than the other). As a result, it is often the case that one constrained efficient allocation is Pareto dominated by another.

We then investigate if there are conditions when all agents are better off under a proportional cost queue than a fixed cost queue, given the same price control. Furthermore, if the price control were removed, would the Walrasian equilibrium be better for some or all of the agents, compared to either type of QRE? The answers may be helpful in setting policy: if event tickets are sold below market price, should we simply raise the price, or maintain the price and allow customers to queue? In the latter case, should we impose a two-per-transaction restriction (which can approximate a proportional cost queue in certain conditions), or allow as many as desired?

Of course, in a general equilibrium model, it is not easy to obtain clean comparisons of equilibrium utilities, even with analytically solvable utility functions; usually numerical examples must be used.[17] However, after computing a number of such examples, one can observe some regularities in the individual welfare consequences of rationing by queues. In particular, all agents are typically worse off under a *f-QRE* than under either a *ρ-QRE* with the same decreed prices, or under a nearby Walrasian equilibrium. Thus, even though *ρ-QRE* is not constrained efficient and a *f-QRE* is, the latter requires much greater queue time and is thus Pareto dominated.

This is formally demonstrated below for the special case of an economy with two goods and two types of households. As long as the price-controlled good is not a Giffen good, the Walrasian equilibrium (weakly) dominates a proportional cost queue at a lower price, and this in turn (weakly) dominates a fixed cost queue at the same price ceiling.

Proposition 2. *Suppose $n = 2$, $\ell = 1$, and $\succcurlyeq_i$ are strictly convex for all i. Also, suppose there are unique* WE (p^w, x^w)*,* f-QRE (p^C, t^f, x^f)*, and* ρ-QRE (p^C, t^ρ, x^ρ)*, where $p^C < p^w$.*

- *If $\frac{\partial \xi^f_{i1}(p,0)}{\partial p} \leq 0$ for all $p \in [p^C, p^C + t^\rho]$, then $x_i^\rho \succcurlyeq_i x_i^f$ for all i.*
- *If $\frac{\partial \xi^\rho_{i1}(p,0)}{\partial p} \leq 0$ for all $p \in [p^C, p^w + t^\rho]$, then $x_i^w \succcurlyeq_i x_i^\rho$ for all i.*
- *If $\frac{\partial \xi^f_{i1}(p,0)}{\partial p} \leq 0$ for all $p \in [p^C, p^w]$, then $x_i^w \succcurlyeq_i x_i^f$ for all i.*

The Pareto inferiority of *f-QRE* is largely a consequence of income effects and substitution effects. While a proportional cost queue incorporates a mixture of both effects, fixed cost queues rely first on income effects, then by participation effects to return aggregate excess demand to zero. Both systems must reduce net demand among buyers (or net supply among sellers) by the same amount, given that the other side of the market will face no queues and the same decreed price. But to accomplish this entirely through the income effect will require more total queue time, and hence fixed cost queues cause lower equilibrium utility than proportional queues. This intuition guides the proof in Appendix **??**.

[17] Numerical results have greatest interest in models of production, which may be calibrated to match national income and product accounts. Nguyen and Whalley (1990) present a general equilibrium framework with a proportional queue (or endogenous transaction cost, in their terminology), which they calibrate to the Canadian economy. They determined that consumers of all incomes would be negatively affected by price controls.

For the same reasons, fixed cost queues are typically Pareto inferior to a nearby Walrasian equilibrium. The imposition of price controls affect demand through both income and substitution effects, but fixed cost queues must counteract these solely through the income effect. Appendix **??** provides an example of both *WE* and ρ-*QRE* Pareto dominating *f-QRE*.

If all buyers (or sellers) were homogeneous, the preceding argument would suffice; however, heterogeneity seriously complicates the analysis. In particular, each agent may have a different "breaking point" at which the fixed cost queue discourages them from participating in the market. When those with a relatively low breaking point abandon the market, it gives the remaining participants extra space to accommodate their increased demand — and as a result, the income effect may not need to be so drastic to achieve an *f-QRE*.

Even so, heterogeneity alone does not prevent *f-QRE* from being Pareto dominated; to create an example in which some agents gain at the expense of others requires careful construction. If a large number of consumers are each buying a small amount of a price-controlled good, they are likely to be driven out of the market with a relatively short queue. The departure of a multitude of petty buyers can reduce demand enough to balance the increased demand of the few remaining big-time patrons, thus constituting a QRE. In such a case, it is possible for the big-time patrons to strictly benefit from the imposition of the price control. An example in Appendix **??** illustrates this.

Put another way, a fixed cost queue relies more heavily on the extensive margin to restore equilibrium (how many people are still participating), while a proportional cost queue operates on the intensive margin (how much they are buying). With homogenous agents, if *any* fraction of buyers have been discouraged from the market (on the extensive margin), *all* buyers will have the same utility as if that market did not exist. With heterogeneous agents, on the other hand, it is difficult to make a general statement as to whether those who still participate are worse off. It is possible to construct examples in which neither QRE is Pareto superior, but it is also possible to find *f-QRE* that are Pareto dominated.

Again, this result is relevant to policy decisions. Even while acknowledging the inefficiencies created in rationing by queues, price controls are sometimes justified as a form of redistribution. For instance, consumers with lower wages will not sacrifice as much by standing in line, and may thus benefit under a QRE, in spite of lost time in the queue.[18] Indeed, this concept has been supported by some of the models with proportional queue costs, such as Bucovetsky (1984). However, it is also possible that *all* agents will be unambiguously harmed by price controls; moreover, fixed cost queues are more likely to produce this result.

Similarly, the fact that *f-QRE* can be Pareto dominated by ρ-*QRE* under rather com-

[18] This heterogeneity can be captured in this model by varying the preferences of agents with respect to good 0.

mon assumptions is both surprising and interesting. It suggests that a proportional cost queue model may yield very different welfare results than a fixed cost queue, making it highly important that the appropriate specification for a particular environment be used. Ironically, the very marginal distortions that make ρ-*QRE*s constrained inefficient also allow them to correct a market imbalance with less cost to the participants than f-*QRE*s with their income effects.

6 Conclusions

Models of rationing by queues produce remarkably different results depending on how the queue time varies with the quantity purchased. Both fixed cost and proportional cost queues are capable of taking the place of prices in bringing aggregate excess demand back to zero, as shown (under a broad class of price controls and with minimal restrictions on preferences) in the existence proofs of this paper. However, proportional cost queues introduce marginal "price-like" distortions, while fixed-cost queues adjust demand (or supply) by reducing wealth and then participation. This difference has implications for both efficiency and individual outcomes.

In terms of efficiency, both specifications cause obvious waste since the time endowment is being destroyed. Even so, proportional cost queues result in additional inefficiency, since the wedge between buyer and seller prices cause marginal distortions. Thus, even after agents have received their equilibrium allocation and lost the time spent in queues, a social planner could identify Pareto-improving trades. This is not the case in fixed cost queues; like a lump sum tax, the damage is limited to the actual lost endowment. At most, fixed cost queues may distort who participates in a market, which may create some opportunities for Pareto improvements.

In comparisons of individual outcomes, typically fixed cost queues result in lower utility for all agents than both the proportional cost equilibrium at the same prices, or the Walrasian equilibrium. This is because income is more of a blunt tool for adjusting the demand of an agent, while a marginal wedge can give a more focused effect.

In addition to these considerations, the assumption of proportional cost of queues is hard to reconcile with many of the stories that are modelled. In light of these issues, researchers should exercise caution in using this form of queue. This specification does offer technical simplicity; however, the methods used in this paper have not proved to be unwieldy.

Indeed, other areas of economic literature could benefit from these same techniques. Any model with transaction cost frictions may profit from this alternative specification. Indeed, financial transaction costs seem much more likely to be fixed rather than proportional to the number of stock shares transacted, for instance. Health economics might also benefit from this work. Rationing by queues commonly occurs in state-run medical systems, though rather than occurring in a literal queue, the waiting is usually on a list. The delay is still costly, particularly when the awaited procedure would significantly improve quality of life. Hopefully, the proofs presented here to obtain existence and other results may serve as a pattern for application in other fields as well.

This topic offers several interesting extensions. First, black markets are also ruled out in this model, which are undoubtedly of interest since they routinely form, even when made illegal and punished by the state. Black markets have been studied in conjunction with proportional cost queues in work such as Stahl and Aleexev (1985), Sah (1987), Suen (1989), Alexeev (1991), and Polterovich (1993). In many cases, the presence of a resale market enhances the efficiency of queuing, though there are notable exceptions. Resale poses an interesting modeling problem for fixed cost queues, since one person could potentially purchase the entire stock (in the absence of credit constraints) and resell it as a monopolist. However, this neglects the endogenous determination of the waiting time. If large rents await the first customer, there will be enormous competition for who arrives first. In the end, free entry into the market for "procurement by queuing" would dissipate these rents, and possibly make it optimal to accommodate competitors rather than fight them off by arriving earlier.

This leads to the next important assumption to relax: the fact that queue times are taken as given. It is natural to think of agents as strategically deciding how early to arrive in order to obtain a better position in the queue. As the number of agents increases, one would expect the outcome to approach a QRE. This would be similar to allowing individual firms some limited market power in prices, and then increasing the number of firms to converge towards a Walrasian equilibrium. This method may even provide a means of selecting among multiple QRE.

It would also be beneficial to incorporate uncertainty in several ways. In recent decades, few governments have attempted full regimes of price controls; rather, the most common usage has been in response to "emergencies" after supply or demand shocks, such as natural disasters. To more appropriately analyze the impact of anti-"price gouging" laws, the model should incorporate consumer uncertainty about the aggregate endowment (and hence the market prices) they might experience in the next period, which in turn affects their choice of how long to queue.

A final question is how this work might be applied to the formation of queues in the absence of price controls. Concert tickets or newly released products often are in short supply, which encourages would-be consumers to stand in line to ensure they can make a purchase. One wonders why a firm would not raise the product's price in such a situation. Perhaps one could apply similar techniques to a model with production by price-setting firms (perhaps in monopolistic competition), and shed additional light on this interesting scenario.

Appendix

A Proof of Theorem 1

The proofs of Theorems 1 and 2 are closely paralleled; thus, to facilitate comparison, the proofs are divided into segments.

A.1 Domain

We begin by normalizing prices on a unit simplex:

$$\Delta_C^k \equiv \left\{ \bar{p} \in \mathbb{R}_+^{\ell+1} : \sum_{j=0}^{\ell} p_j = 1,\; p_j \geq \frac{1}{k} \text{ and } p_0\, p_j^F \leq p_j \leq p_0\, p_j^C \right\}. \tag{8}$$

With strictly monotone and convex preferences, if the price of a good approaches 0, excess demand for one of the goods typically approaches infinity. To keep demand well defined, we restrict prices away from zero, assuming k to be sufficiently large so that Δ_C^k is non-empty. Convexity and closedness immediately follow.

Because queue costs are proportional to quantities purchased just as dollar prices are, we can represent the "total price" of goods (dollars and time) in another simplex:

$$\Delta_T^k \equiv \left\{ \bar{\tau} \in \mathbb{R}_+^{\ell+1} : \sum_{j=0}^{\ell} \tau_j = 1 \text{ and } \tau_j \geq \frac{1}{k} \text{ for all } j \right\}. \tag{9}$$

One can extract the proportional queue cost from the total price (in units comparable to the dollar prices) using $t_j(\bar{p}, \bar{\tau}) \equiv \frac{p_0 \tau_j}{\tau_0} - p_j$. Note that τ is not a literal price, since the waiting cost only applies to one side of the market at a time, but this will be accounted for in our demand correspondence.

To ensure that queues will only occur in markets where a price control is binding, we define a correspondence $T^k : \Delta_C^k \rightrightarrows \Delta_T^k$,which selects a subset of the total price simplex for a given dollar price. In particular, a queue may only be positive if the dollar price is near the price ceiling, or negative if it is near the price floor. The precise bounds are chosen to ensure that T^k is a continuous correspondence. As a subset of Δ_T^k, total prices in T^k are still bounded away from 0.

$$T^k(\bar{p}) \equiv \left\{ \bar{\tau} \in \Delta_T^k : \begin{array}{ll} \tau_j \in [0,1] & \text{if } p_j^F = \dfrac{p_j}{p_0} = p_j^C \\ \tau_j \in \left[k\left(\dfrac{p_j}{p_0} - p_j^F\right)\dfrac{\tau_0 p_j}{p_0}, \dfrac{\tau_0 p_j}{p_0} \right] & \text{if } \dfrac{p_j}{p_0} \le p_j^F + \dfrac{1}{k} < p_j^C \\ \tau_j \in \left[\dfrac{\tau_0 p_j}{p_0}, 1 - k\left(p_j^C - \dfrac{p_j}{p_0}\right)\left(1 - \dfrac{\tau_0 p_j}{p_0}\right) \right] & \text{if } p_j^F < p_j^C - \dfrac{1}{k} \le \dfrac{p_j}{p_0} \\ \tau_j = \dfrac{\tau_0 p_j}{p_0} & \text{if } p_j^F + \dfrac{1}{k} < \dfrac{p_j}{p_0} < p_j^C - \dfrac{1}{k} \end{array} \right\}$$

In the first case, the price control in market j is always binding, so any queue is permitted. In the fourth, the price is well inside the interior, so the relative price between leisure and good j must be the same in $\bar{\tau}$ as it is in $\bar{p}$. In the second and third cases, the price is at or near the floor (*resp.*, ceiling), so the relative total price can be less than (*resp.*, more than) the relative dollar price.

A.2 Correspondences

Next, we construct the correspondences to be used in the fixed point argument. For each k, there exists a bound $s^k > 0$ such that for all j and i, $-s^k \le \bar{\eta}_{ij}(\bar{p}, t(\bar{p}, \bar{\tau})) \le s^k$ for any $\bar{p} \in \Delta_C^k$ and any $\bar{\tau} \in T^k(\bar{p})$. Let $\bar{Z}^k = [-s^k, s^k]^{\ell+1}$. We then define the following correspondences:

$\zeta^k : \Delta_C^k \times \Delta_T^k \rightrightarrows \bar{Z}^{kN}$, where for all i,

$$\zeta_i^k(\bar{p}, \bar{\tau}) \equiv \Big\{ \bar{z} \in \bar{Z}^k : \exists \bar{x}_i \in X_i \text{ s.t. } \bar{x}_i \in \bar{\xi}_i(\bar{p}, t(\bar{p}, \bar{\tau})),\ z_i \le x_i - \omega_i \\ \text{and } z_{i0} \le x_{i0} - \omega_{i0} + Q^\rho(z_i, t(\bar{p}, \bar{\tau})) \Big\} \tag{10}$$

ζ_i^k is similar to the excess demand function for agent i; the only alteration is that excess demand for the numeraire, z_{i0}, includes not only consumed leisure but also time devoted to queuing.

$\mu^k : \bar{Z}^{kN} \rightrightarrows \Delta_C^k$, where

$$\mu^k((\bar{z}_i)_{i=1}^N) \equiv \left\{ \bar{p} \in \Delta_C^k : \ \bar{p} \cdot \sum_{i=1}^N \bar{z}_i \ge \hat{p} \cdot \sum_{i=1}^N \bar{z}_i,\ \forall \hat{p} \in \Delta_C^k \right\} \tag{11}$$

μ is an adaptation of the typical correspondence for prices, placing the maximum weight possible on whichever good(s) are in highest demand. Note that the demand for time spent in queues is included in this computation.

$\psi^k : \Delta_C^k \times \bar{Z}^{kN} \rightrightarrows \Delta_T^k$, where

$$\psi^k(\bar{p}, (\bar{z}_i)_{i=1}^N) \equiv \left\{ \bar{\tau} \in T^k(\bar{p}) : \ \bar{\tau} \cdot \sum_{i=1}^N \bar{z}_i \geq \hat{\tau} \cdot \sum_{i=1}^N \bar{z}_i, \ \forall \hat{\tau} \in T^k(\bar{p}) \right\} \tag{12}$$

ψ mimics μ, placing the highest "total price" on goods that are in highest excess demand. Since Δ_T^k is not constrained by price controls, ψ will result in more extreme prices, which reflect queues for buyers or for sellers, but only in regulated markets (due to the restrictions imposed by T^k).

The fixed point correspondence is thus defined as: $\phi^k : \Delta_C^k \times \Delta_T^k \times \bar{Z}^{kN} \rightrightarrows \Delta_C^k \times \Delta_T^k \times \bar{Z}^{kN}$

$$\phi^k\left(\bar{p}, \bar{\tau}, (\bar{z}_i)_{i=1}^N\right) = \mu^k\left((\bar{z}_i)_{i=1}^N\right) \times \psi^k\left(\bar{p}, (\bar{z}_i)_{i=1}^N\right) \times \prod_{i=1}^N \zeta_i^k(\bar{p}, \bar{\tau}) \tag{13}$$

The domain of ϕ^k is non-empty (for k sufficiently large), compact, and convex. In addition, ϕ^k is non-empty valued, convex-valued, and upper hemicontinuous for the whole domain. (The latter property comes from the continuity of the dot product and the upper hemicontinuity of the individual demand correspondences). Thus Kakutani's Fixed Point Theorem applies: there exists a $\left(\bar{p}^{k*}, \bar{\tau}^{k*}, (\bar{z}_i^{k*})_{i=1}^N\right)$ such that $\left(\bar{p}^{k*}, \bar{\tau}^{k*}, (\bar{z}_i^{k*})_{i=1}^N\right) \in \phi\left(\bar{p}^{k*}, \bar{\tau}^{k*}, (\bar{z}_i^{k*})_{i=1}^N\right)$. Let $t^{k*} \equiv t\left(\bar{p}^{k*}, \bar{\tau}^{k*}\right)$.

A.3 Excess demand in the limit

Since $\bar{z}_i^{k*}$ is the aggregate excess demand for agent i (including time spent in queues), we can apply Walras' law (from Lemma 2) to each agent type, and summing across all types, obtain $\bar{p}^{k*} \cdot \bar{z}^{k*} = 0$, where $\bar{z}^{k*} = \sum_{i=1}^N \bar{z}_i^{k*}$. Also, from μ, $\bar{p}^{k*} \cdot \bar{z}^{k*} \geq \hat{p} \cdot \bar{z}^{k*}$ for all $\hat{p} \in \Delta_C^k$. Thus $\hat{p} \cdot \bar{z}^{k*} \leq 0$ for all $\hat{p} \in \Delta_C^k$.

Next we show that the sequence $\{\bar{z}^{k*}\}$ is contained in a compact set. Define the largest aggregate endowment among the various goods as $b = \max_{j=0}^{\ell} \sum_{i=1}^N \omega_{ij}$, and define the set:

$$\Pi^k = \left\{ \bar{z} \in \mathbb{R}^{\ell+1} : \ \bar{p} \cdot \bar{z} \leq 0, \forall \bar{p} \in \Delta_C^k \text{ and } z_j \geq -b, \forall j \right\}. \tag{14}$$

Observe that $\Pi^k \supset \Pi^{k+1}$ and $\bar{z}^{k*} \in \Pi^k$, and thus $\bar{z}^{k'*} \in \Pi^k$ for all $k' \geq k$. Consider $\bar{z} \in \Pi^k$, and for any good j where $\bar{z}_j > 0$, choose a price $\bar{p}(j) \in \Delta_C^k$ such that the price for good j is as low as possible: $p_j(j) \leq \hat{p}_j$ for all $\hat{p} \in \Delta_C^k$. This could be as low as $\frac{1}{k}$; however, price controls may constrain j's price to be higher: $p_j(j) \geq \frac{1}{k}$. The sum of other prices must be $\sum_{j' \neq j} p_{j'}(j) \leq \frac{k-\ell-1}{k}$. Thus, since $\bar{p}(j) \cdot \bar{z} \leq 0$,

$$p_j(j)\, z_j \leq -\sum_{j' \neq j} p_{j'}(j)\, z_{j'} \quad \Longrightarrow \quad \frac{1}{k}\, z_j \leq b \sum_{j' \neq j} p_{j'}(j) \quad \Longrightarrow \quad z_j \leq b\,(k - \ell - 1)$$

So for each $\bar{z} \in \Pi^k$ and each j, $-b \leq z_j \leq b \cdot (k - \ell - 1)$. Thus, for all $k' \geq k$, $\bar{z}^{k'*} \in [-b, b \cdot (k - \ell - 1)]^{\ell+1}$. (The fact that this constructed bound grows with k is unimportant; it merely establishes Π^k is bounded and therefore bounds all of its subsets, $\Pi^{k'}$). Thus, we may restrict ourselves to some subsequence of k for which $\bar{z}^{k*}$ converges to some $\bar{z}^*$. Since each individual has a lower bound on their excess demand, this implies that each individual's $\bar{z}_i^{k*}$ is similarly convergent.

Δ_C^0 and Δ_T^0 are both compact and contain the sequences $\{\bar{p}^{k*}\}$ and $\{\bar{\tau}^{k*}\}$, respectively; therefore,these sequences have a subsequence (to which we can further restrict ourselves) which converges to some $\bar{p}^*$ and $\bar{\tau}^*$. By definition, the limit price vector will satisfy any restrictions imposed by price controls.

A.4 Non-zero numeraire price

Suppose that $p_0^* = 0$. Note that this could not occur if all of the other goods have a price ceiling $p_j^C < +\infty$: ceilings are specified in terms of good 0, so as p_0 is reduced, eventually all other prices would hit their ceilings, at which point further reduction in p_0 would result in prices not being on the simplex. Indeed, for $p_0^{k*} \underset{k\to\infty}{\longrightarrow} 0$ to occur, there must be some good j such that $p_j^C = +\infty$ and $p_j^{k*} > p_j^F \cdot p_0^{k*}$ for k sufficiently large. Let J be the set of all such goods. This, in turn, requires $t_j^{k*} = 0$ for large enough k, from the definition of $T^k(\bar{p}^{k*})$.

Because the aggregate endowment is positive for all goods, there exists some agent i such that $p_j^{k*} \cdot \omega_{ij}$ is bounded away from 0 for all k. Since there is no queue for buyers or sellers of $j \in J$, this ensures that agent i can sell good j and thus buy at least good 0 and possibly other goods (depending on their queues) as their prices approach 0. If $z_{ij}^* > -\omega_{ij}$, then we can choose an $\epsilon > 0$ such that for all k sufficiently large, $z_{ij}^{k*}(\epsilon) \equiv z_{ij}^{k*} - \epsilon \cdot \frac{p_0^{k*}}{p_j^{k*}} > -\omega_{ij}$. Let $z_{i0}^{k*}(\epsilon) \equiv z_{i0}^{k*} + \epsilon$ and $z_{ij'}^{k*}(\epsilon) \equiv z_{ij'}^{k*}$ for $j' \neq j$. Note that $\bar{z}_i^{k*}(\epsilon)$ is affordable under $\bar{p}^{k*}$ and t^{k*}, since $\bar{z}_i^{k*}$ is. Moreover, $z_i^{k*}(\epsilon) \underset{k\to\infty}{\longrightarrow} z_i^*$, and $z_{i0}^{k*}(\epsilon) \underset{k\to\infty}{\longrightarrow} z_{i0}^* + \epsilon$. Due to strictly monotone preferences, $\bar{z}_i^*(\epsilon) + \omega_i \succ_i \bar{z}_i^* + \omega_i$, and by the continuity of preferences, this means there exists some k for which $\bar{z}_i^{k*}(\epsilon) + \omega_i \succ_i \bar{z}_i^{k*} + \omega_i$. This contradicts the fact that $\bar{z}_i^{k*}$ comes from solving the consumer's maximization problem.

Suppose instead that for all i, $z_{ij}^* = -\omega_{ij}$ for all goods $j \in J$. Thus, for some $c > 0$, $\sum_i \sum_j p_j^{k*} z_{ij}^{k*} \underset{k\to\infty}{\longrightarrow} -c < 0$, which necessitates $\sum_i \sum_{j' \notin J} p_{j'}^{k*} z_{ij'}^{k*} \underset{k\to\infty}{\longrightarrow} c$. But since $p_{j'}^{k*} \leq p_0^{k*} p_{j'}^C \underset{k\to\infty}{\longrightarrow} 0$ for all $j' \notin J$, $z_{j'}^{k*}$ would have to be unbounded, which contradicts. Thus, $p_0^* > 0$.

A.5 Non-zero prices

By the continuity of the dot product operator, $\bar{p}^* \cdot \bar{z}^* = 0 \geq \hat{p} \cdot \bar{z}^*$ for all $\hat{p} \in \Delta_C^0$. This rules out the possibility that $\bar{z}_j^* \geq 0$ for all j with strict inequality for some goods, such as j'. If this were to occur, ϕ would set $p_{j'}^*$ to be strictly positive for at least one of the j' goods, making the cross product $\bar{p}^* \cdot \bar{z}^*$ strictly positive.

Next, we must rule out the possibility that a non-numeraire price approaches 0. Suppose that $p_j^* = 0$ for some $j \in \{1, \ldots, \ell\}$. Since $p_0^* > 0$, this can only occur for goods with $p_j^F = 0$, and would imply that $p_j^{k*} < p_0^{k*} p_j^C$ for k sufficiently large. Thus, $t_j^{k*} = 0$.

We can then repeat the strategy used with p_0^*: every agent i has strictly positive wealth under $\bar{p}^*$ due to the time endowment. As long as he has not sold all of his endowment in the limit, he would be able to increase consumption of good j by $\epsilon > 0$, by sacrificing an ever decreasing amount of good 0 consumption. In the limit, this alternate bundle will be strictly preferred, and thus, for some k, the alternate bundle is both affordable and strictly preferred to $\bar{z}_i^{k*} + \omega_i$, which contradicts.

Suppose then that for all goods $j' \neq j$, either $z_{ij'}^{k*} = -\omega_{ij'}$ (including good 0), or $z_{ij'}^{k*} = 0$ and $t_{j'} < 0$. This means that good 0 must fall into the former group with the full endowment being sold (since the latter condition cannot apply), and the proceeds from that sale are not being used on any other goods j'. By Walras' law, they must be used on good j. Recall that $t_j^{k*} = 0$, so queues cannot be preventing increased purchases of j. This would ensure that z_{ij}^{k*} is unbounded, which contradicts.

Thus we know that $\bar{p}^* \gg 0$, which means that $T^0(\bar{p}^*)$ and $\zeta_i^0(\bar{p}^*, t^*)$ are well defined. (Note that there is no problem if τ^* produces $t_j^* = -p_j^*$ or $t_j^* = \infty$; this would make for a very low (or a very high) total price only if the agent wants to sell (or buy) the good, discouraging them from doing so. The other side of the market will still face positive prices.) Since ξ is upper hemicontinuous over strictly positive prices, $\bar{z}_i^* \in \zeta_i^0(\bar{p}^*, t^*)$. Similarly, $\bar{\tau}^* \cdot \bar{z}^* \geq \hat{\tau} \cdot \bar{z}$ for all $\hat{\tau} \in T^0(\bar{p}^*)$.

Suppose $\bar{z}_j^* > 0$ for some j (and hence $\bar{z}_k^* < 0$ for some k). Let j identify the largest component. Then for all j' where $\bar{z}_{j'}^* < \bar{z}_j^*$, either $p_{j'}^* = 0$ or, if $p_{j'}^* = p_0^* p_{j'}^F$, then $\tau_{j'}^* = 0$. In either case, there will be no one willing to sell any of this good, regardless of the official price, since the effective price will be zero. Indeed, for all i, $\bar{z}_{ij'}^* \geq 0$, and so $\bar{z}_{j'}^* \geq 0$ for all j'. As j is the largest component of $\bar{z}$, it must be that $p_j^* > 0$, and hence $\bar{p}^* \cdot \bar{z}^* > 0$, which contradicts. Thus, $\bar{z}_j^* \leq 0$ for all j.

Now consider if $\bar{z}_j^* < 0$ for some j and all goods j' having $\bar{z}_{j'}^* \leq 0$ (and, indeed, equal to zero for some j' to avoid a negative cross product of demand and prices). If some good j had a strictly positive price floor, then $p_j \bar{z}_j^*$ will be a negative component in the cross product (with no positive component to compensate). Thus it must be that $p_{j'}^F = 0$ for all goods with $\bar{z}_j^* < 0$. Then $p_j^* = 0$ and $\tau_j^* = 0$ if $j \in J$, since j is not the largest component of excess demand. But this contradicts with $p_j^* > 0$.

Thus $z^* = 0$, and is supported by equilibrium prices $\bar{p}^*$ and queues t^*. By construction of $T^0(\bar{p}^*)$, $t_j^* > 0$ only when $p_j^* = p_0^*\, p_j^C$, and $t_j^* < 0$ only when $p_j^* = p_0^*\, p_j^F$.

B Proof of Theorem 2

B.1 Domain

The domain of prices, Δ_C^k is the same as in Section A.1 — which is to say that prices are normalized on a simplex, bounded away from zero by $\frac{1}{k}$, and permissible under the price control. The domain of queue times, on the other hand, cannot be represented on a simplex. Instead, we construct a correspondence $T^k(\bar{p})$ that forms a hyperrectangle based on the dollar price. $T^k(\bar{p})$ forces the queue to zero unless prices are near a bound.

To begin, we identify a quantity of leisure too large for any individual to afford — at any admissible price, or at a particular price $\bar{p} \in \Delta_C^k$, respectively:

$$M^k = 1 + \max_{i=1} \max_{\bar{p} \in \Delta_C^k} \frac{\bar{p} \cdot \bar{\omega}_i}{p_0} \qquad\qquad M(\bar{p}) = 1 + \max_{i=1} \frac{\bar{p} \cdot \bar{\omega}_i}{p_0}$$

Note that $M(\bar{p}) \leq M^k$ for any $\bar{p} \in \Delta_C^k$.

If $t_j \geq M(\bar{p})$ (*resp.*, $t_j \leq -M(\bar{p})$), then $B_j(\bar{p}, t; \bar{\omega}_i) \subset \{x_i : \ x_{ij} \leq \omega_{ij}\}$ (*resp.*, $\subset \{x_i : \ x_{ij} \geq \omega_{ij}\}$) for all i, because the queuing time cost is too great to purchase (sell) any of j. Under such a t, for any $\bar{x} \in \xi(\bar{p}, t)$, $\sum_{i=1}^N (x_{ij} - \omega_{ij}) \leq 0$ (≥ 0). Define $T^k \equiv [-M^k, M^k]^\ell$ as the domain of queue lengths. For any particular set of prices, we further limit admissible queue lengths according to the following (which is specified so as to be a continuous correspondence on Δ_C^k):

$$T^k(\bar{p}) \equiv \left\{ t \in \mathbb{R}^\ell : \begin{array}{ll} t_j \in [-M(\bar{p}), M(\bar{p})] & \text{if } p_j^F = \dfrac{p_j}{p_0} = p_j^C \\ t_j \in \left[\left(k \left(\dfrac{p_j}{p_0} - p_j^F \right) - 1 \right) M(\bar{p}), 0 \right] & \text{if } \dfrac{p_j}{p_0} \leq p_j^F + \dfrac{1}{k} < p_j^C \\ t_j \in \left[0, \left(1 - k \left(p_j^C - \dfrac{p_j}{p_0} \right) \right) M(\bar{p}) \right] & \text{if } p_j^F < p_j^C - \dfrac{1}{k} \leq \dfrac{p_j}{p_0} \\ t_j = 0 & \text{if } p_j^F + \dfrac{1}{k} < \dfrac{p_j}{p_0} < p_j^C - \dfrac{1}{k} \end{array} \right\}$$

B.2 Correspondences

We use the same $\bar{Z}^k$ as a bound for excess demand. For this proof, though, the aggregate excess demand correspondence requires additional tracking of queue costs: since various agents of type i may consume different bundles (though equal in utility), we provide the means to identify any convex combination of bundles[19]. It is necessary to track which bundles were used to obtain a particular aggregate excess demand in order to calculate the aggregate queue time. We define $A = [-1, 1]^\ell$, then define the following correspondences:

$\zeta^k : \Delta_C^k \times T^k \rightrightarrows (\bar{Z}^k \times A)^N$, where for all i,

$$\zeta_i^k(\bar{p}, t) \equiv \Bigg\{ (\bar{z}_i, \alpha_i) \in \bar{Z}^k \times A : \exists (\bar{x}_i^m, \beta_m)_{m=1}^{\ell+2} \in X_i^{\ell+2} \times [0,1]^{\ell+2} \text{ s.t.}$$
$$\bar{x}_i^m \in \bar{\xi}_i(\bar{p}, t)\ \forall m, \quad \sum_{m=1}^{\ell+2} \beta_m = 1, \quad \bar{z}_i = \sum_{m=1}^{\ell+2} \beta_m (\bar{x}_i^m - \bar{\omega}_i), \text{ and} \tag{15}$$
$$\alpha_{ij} = \sum_{m=1}^{\ell+2} \beta_m \Big(I(\bar{x}_{ij}^m > \omega_{ij}) I(t_j > 0) - I(\bar{x}_{ij}^m < \omega_{ij}) I(t_j < 0) \Big) \Bigg\}$$

Note that $\alpha_{ij} > 0$ (< 0) denotes the fraction of agents of type i who choose to queue so that they can purchase (sell) good j. Unlike ζ^k in Equation 10, excess demand for the numeraire does not include queue time.

$\mu^k : T^k \times (\bar{Z}^k \times A)^N \rightrightarrows \Delta_C^k$, where

$$\mu^k(t, (\bar{z}_i, \alpha_i)_{i=1}^N) \equiv \Bigg\{ \bar{p} \in \Delta_C^k : \sum_{i=1}^N (\bar{p} \cdot \bar{z}_i + p_0 (t \cdot \alpha_i)) \geq \sum_{i=1}^N (\hat{p} \cdot \bar{z}_i + \hat{p}_0 (t \cdot \alpha_i)), \forall \hat{p} \in \Delta_C^k \Bigg\} \tag{16}$$

The only difference from Equation 11 is that demand for time spent in queues must be explicitly added into the computation, since it is not included in z_{i0}.

The definition of ψ is the same in this proof as in Equation 12, differing only because $T^k(\bar{p})$ is distinct for fixed cost queues. [20] Note that *any* good in excess demand (or supply) will have the maximum (minimum) queue time imposed — unlike μ, which only puts the greatest price weight on the good with largest excess demand. The caveat is that queue

[19] The Caratheodory Convexity Theorem provides that for the convex hull of a non-empty set in an $\ell+1$-dimensional vector space, any vector can be formed from the convex combination of at most $\ell+2$ vectors of the set.

[20] Note that an element of T^k is t rather than τ, meaning that it can be immediately interpreted as the queue cost rather than requiring translation from a total price.

times are limited to 0 unless prices are close to a ceiling or floor. (The purpose of allowing queues near the price control is so that this maximization may occur over a continuous correspondence of p, thus ensuring that ψ^k is upper hemicontinuous.)

The fixed point correspondence is thus defined as: $\phi^k : \Delta_C^k \times T^k \times (\bar{Z}^k \times A)^N \rightrightarrows \Delta_C^k \times T^k \times (\bar{Z}^k \times A)^N$ as:

$$\phi^k(\bar{p}, t, (\bar{z}_i, \alpha_i)_{i=1}^N) = \mu^k(t, (\bar{z}_i, \alpha_i)_{i=1}^N) \times \psi^k(\bar{p}, (z_i)_{i=1}^N) \times \prod_{i=1}^N \zeta_i^k(\bar{p}, t) \tag{17}$$

As before, Kakutani's Fixed Point Theorem applies: there exists a $(\bar{p}^{k*}, t^{k*}, (\bar{z}_i^{k*}, \alpha_i^{k*})_{i=1}^N)$ such that $(\bar{p}^{k*}, t^{k*}, (\bar{z}_i^{k*}, \alpha_i^{k*})_{i=1}^N) \in \phi(\bar{p}^{k*}, t^{k*}, (\bar{z}_i^{k*}, \alpha_i^{k*})_{i=1}^N)$.

B.3 Excess demand in the limit

If we apply Walras' law as in Section A.3, we get $\bar{p}^{k*} \cdot \bar{z}^{k*} + \bar{p}_0^{k*} \sum_{i=1}^N \left(t^{k*} \cdot \alpha_i^{k*}\right) = 0$. Also, from μ, $\bar{p}^{k*} \cdot \bar{z}^{k*} + p_0^{k*} \sum_{i=1}^N \left(t^{k*} \cdot \alpha_i^{k*}\right) \geq \hat{p} \cdot \bar{z}^{k*} + \hat{p}_0^* \sum_{i=1}^N \left(t^{k*} \cdot \alpha_i^{k*}\right)$ for all $\hat{p} \in \Delta_C^k$. Since $t_j^{k*}\, \alpha_{ij}^{k*} \geq 0$ for all i, j, and k, this implies that $\hat{p} \cdot \bar{z}^{k*} \leq 0$ for all $\hat{p} \in \Delta_C^k$.

The proof that $\bar{z}_i^{k*}$ converges to some $\bar{z}_i^*$ is identical to that in Section A.3.

Δ_C^0 and $A^{\ell \cdot N}$ are both compact and contain the sequences $\{p^{k*}\}$ and $\{\alpha^{k*}\}$, respectively; therefore,these sequences have a subsequence (to which we can further restrict ourselves) which converges to some p^* and α^*. By definition, the limit price vector will satisfy any restrictions imposed by price controls.

B.4 Non-zero numeraire price

This proof proceeds identically to Section A.4 for the first two paragraphs, ruling out the possibility that $p_0^* = 0$ and $z_{ij}^* > -\omega_{ij}$ for some good $j \in J$ (*i.e.* those goods without price ceilings).

Suppose instead that $p_0^* = 0$ and $z_{ij}^* = -\omega_{ij}$ for all goods $j \in J$. Thus, for some $c > 0$, $\sum_i \sum_j p_j^{k*} z_{ij}^{k*} \underset{k\to\infty}{\longrightarrow} -c < 0$, which necessitates $\sum_i \sum_{j' \notin J} p_j^{k*} z_{ij'}^{k*} + p_0^{k*} \alpha_{ij'}^{k*} t_{j'}^{k*} \underset{k\to\infty}{\longrightarrow} c$. Note that $p_j^{k*} \leq p_0^{k*} p_j^C \underset{k\to\infty}{\longrightarrow} 0$ for all $j \notin J$, and since z_j^{k*} has been shown to be bounded, the first term in the sum will approach zero.

Alternatively, the budget could be increasingly devoted to queue time — that is, the increasing purchasing power is consumed in purchasing additional time. By construction, $\alpha_{ij'}^{k*} \leq 1$, so it would have to be that $t_j^{k*} \underset{k\to\infty}{\longrightarrow} +\infty$. But if this were to occur, the numeraire would eventually be larger than any component of $\bar{z}^{k*}$. Thus, μ would have to set p_0^{k*} to the highest value possible, contradicting the assumption that p_0^{k*} is approaching 0 (or contradicting the fixed point property). Hence $p_0^* > 0$.

B.5 Non-zero prices

Since the price of time converges to be non-zero, $M(\bar{p}^{k*})$ will converge (even though M^k increases with k). To be part of a sequence of fixed points of ϕ, the sequence $\{t^{k*}\}$ must be contained in $[-M(\bar{p}^{k*}), M(\bar{p}^{k*})]^\ell$. Thus, we may take a subsequence (of the subsequence used for prices) that will converge to some $t^* \in [-M(\bar{p}^*), M(\bar{p}^*)]^\ell$.

Next, we must rule out the possibility that any non-numeraire price approaches 0. Suppose that $p_j^* = 0$ for some $j \in \{1, \ldots, \ell\}$. Since $p_0^* > 0$, this can only occur for goods with $p_j^F = 0$, and would imply that $p_j^{k*} < p_0^{k*} p_j^C$ for k sufficiently large. Thus, $t_j^{k*} = 0$.

We can then repeat the strategy used with p_0^*: some agent i has strictly positive wealth under $\bar{p}^*$. As long as he has not sold all of his endowment or queues are not prohibitively long (*e.g.* there exists some good $j' \in \{0, 1, \ldots, \ell\}$ such that $z_{ij'}^{k*} > -\omega_{ij}$ and if $t_{ij'} < 0$ then $z_{ij'}^{k*} \neq 0$), he would be able to increase consumption of good j by $\epsilon > 0$, by sacrificing an ever decreasing amount of good j' consumption. In the limit, this alternate bundle will be strictly preferred, and thus, for some k, the alternate bundle is both affordable and strictly preferred to $\bar{z}_i^{k*} + \omega_i$, which contradicts.

Suppose then that for all goods $j' \neq j$, either $z_{ij'}^{k*} = -\omega_{ij'}$ (including good 0), or $z_{ij'}^{k*} = 0$ and $t_{j'} < 0$. This means that good 0 must fall into the former group with the full endowment being sold (since the latter condition cannot apply), and the proceeds from that sale are not being used on any other goods j'. By Walras' law, they must be used on good j. Recall that $t_j^{k*} = 0$, so queues cannot be preventing increased purchases of j. This would ensure that z_{ij}^{k*} is unbounded, which contradicts.

Thus we know that $\bar{p}^* \gg 0$, which means that $T^0(\bar{p}^*)$ and $\zeta_i^0(\bar{p}^*, t^*)$ are well defined. Moreover, since ξ is upper hemicontinuous over strictly positive prices, $(\bar{z}_i^*, \alpha_i^*) \in \zeta_i^0(\bar{p}^*, t^*)$. μ^0 is also well defined and continuous as $k \to +\infty$, so $\bar{p}^* \in \mu^0(t^*, (\bar{z}_i^*, \alpha_i^*)_{i=1}^N)$. Also, $\bar{p}^* \cdot \bar{z}^* + p_0^* \sum_{i=1}^N (t^* \cdot \alpha_i^*) = 0$.

Suppose $z_j^* < 0$ for some j. Thus $z_j^{k*} < 0$ for large enough k. Note also that there must be some good j' with positive aggregate demand $z_{j'}^* > 0$, or good 0 with $z_0^* + \sum_{i=1}^N (t^* \cdot \alpha_i^*) > 0$. Otherwise, $\bar{p}^* \cdot \bar{z}^* + p_0^* \sum_{i=1}^N (t^* \cdot \alpha_i^*) < 0$. Then under μ^k and μ^0, $p_j^{k*} = p_0^{k*}\, p_j^F$ and $p_j^* = p_0^*\, p_j^F$. If $p_j^F = 0$, this contradicts with $\bar{p}^* \gg 0$. (The same reasoning rules out $z_0^* + \sum_{i=1}^N (t^* \cdot \alpha_i^*) < 0$.) If $p_j^F > 0$, then $t_j^{k*} = -M(\bar{p}^{k*})$. But since no one can afford to sell the good and throwing it away would not be optimal, $z_j^{k*} \geq 0$, which contradicts.

Thus, $z_j^* \geq 0$ for all $j \in \{1, \ldots, \ell\}$, and $z_0^* + \sum_{i=1}^N (t^* \cdot \alpha_i^*) \geq 0$. But if any of these were strict inequalities, $\bar{p}^* \cdot \bar{z}^* + p_0^* \sum_{i=1}^N (t^* \cdot \alpha_i^*) > 0$. Thus, $z_j^* = 0$, so the associated allocation together with p^* and t^* constitute a QRE. Indeed, if $p_0\, p_j^F < p_j^* < p_0\, p_j^C$ then $t_j^* = 0$, by the properties of $T^0(p^*)$. Similarly, if $t_j^* > 0$ or $t_j^* < 0$, then $p_j^* = p_0\, p_j^C$ or $p_j^* = p_0\, p_j^F$, respectively.

C Proof of Theorem 3

Suppose $\left((\bar{x}_i^*)_{i=1}^N, p^*, t^*\right)$ is a QRE, but not an Essential QRE; so there exists some k such that $t_k^* \neq 0$ and for all i and all $\bar{x}_i \in \xi_i(p^*, t^*)$, $t_k^* \cdot (x_{ik} - \omega_{ik}) \leq 0$. We will proceed as if there were only one such k; but if there were several, we could repeat the procedure in each dimension. Similarly, examine the case when $t_k^* > 0$ and $x_{ik} \leq \omega_{ik}$ for all i without loss of generality.

Let $V_{ik} = \{\bar{x}_i \in \mathbb{R}^{\ell+1} :\ x_{ik} \leq \omega_{ik}\}$. Define $t(\gamma)$ as $t_j(\gamma) = t_j^*$ for $j \neq k$ and $t_k(\gamma) = \gamma \cdot t_k^*$. For any $\gamma, \beta \geq 0$, $B(p^*, t(\gamma); \bar{\omega}_i) \cap V_{ik} = B(p^*, t(\beta); \bar{\omega}_i) \cap V_{ik}$. This is to say that the buyer's queue length will not affect the budget of an agent who does not choose to buy that good. Moreover, since $\bar{x}_i^* \in V_{ik}$, for any $\gamma \geq 0$, then $\bar{x}_i^* \in B(p^*, t(\gamma); \bar{\omega}_i) \cap V_{ik}$.

Let $U_i = \{\bar{x} \in \mathbb{R}^{\ell+1} :\ \bar{x}_i \succcurlyeq_i \bar{x}_i^*\}$, which is non-empty, closed, and convex. For each agent, identify the set of expenditure-minimizing bundles that weakly improve utility as $E_i \equiv \{\bar{x}_i \in U_i :\ x_{i0} + p^* \cdot x_i + Q(x_i - \omega_i, t(0)) \leq x'_{i0} + p^* \cdot x'_i + Q(x'_i - \omega_i, t(0)),\ \forall \bar{x}'_i \in U_i\}$. These sets are well-defined, as the minimization of a continuous function over a non-empty, compact set. Note that while we have shut down the fixed cost for good k, all other queue constraints are still in effect.

Since $\bar{x}_i^* \in U_i \cap B(p^*, t(0); \bar{\omega}_i)$, any $\bar{x}_i \in E_i$ must cost no more than $\bar{x}_i^*$. Thus, we can define a unique $\gamma_i \geq 0$ for each i such that for any $\bar{x}'_i \in E_i$,

$$x'_{i0} + p^* \cdot x'_i + Q(x'_i - \omega_i, t(0)) + \gamma_i\, t_k^* = x_{i0}^* + p^* \cdot x_i^* + Q(x_i^* - \omega_i, t(0)).$$

Moreover, $\bar{x}'_i \sim_i \bar{x}_i^*$ (e.g. $\bar{x}'_i$ is on the boundary of U_i; otherwise, $\bar{x}'_{i0}$ could be reduced by some $\epsilon > 0$, keeping it in U_i and yet lowering expenditure). As an immediate result, if $\gamma_i = 0$, then $\bar{x}_i^* \in \xi_i(p^*, t(0))$.

Suppose that $\bar{x}'_i \in E_i$ and $x'_{ik} \leq \omega_{ik}$. Since $\bar{x}'_i \in B(p^*, t(0); \bar{\omega}_i) \cap V_{ik}$, then $\bar{x}'_i \in B(p^*, t^*; \bar{\omega}_i) \cap V_{ik}$. If $x'_{i0} + p^* \cdot x'_i + Q(x'_i - \omega_i, t(0)) < x_{i0}^* + p^* \cdot x_i^* + Q(x_i^* - \omega_i, t(0))$, then x'_{i0} could be increased by some $\epsilon > 0$, keeping it affordable under $B(p^*, t^*; \bar{\omega}_i)$ but making it strictly preferred over $\bar{x}_i^*$. This would contradict $\bar{x}_i^* \in \xi_i(p^*, t^*)$. Hence, it is only possible for $\gamma_i > 0$ if there is some $\bar{x}'_i \in E_i$ where $x'_{ik} > \omega_{ik}$.

Consider the case where $\gamma_i > 0$, $\bar{x}'_i \in E_i$ and $x'_{ik} > \omega_{ik}$. By construction, $\bar{x}'_i$ is on the boundary of $B(p^*, t(\gamma_i); \bar{\omega}_i)$. Suppose there exists some $\bar{x}_i'' \in B(p^*, t(\gamma_i); \bar{\omega}_i)$ such that $\bar{x}_i'' \succ_i \bar{x}_i'$. Since $\bar{x}_i' \in E_i$ and $\bar{x}_i'' \in U_i$, it must be that $x_{i0}'' + p^* \cdot x_i'' + Q(x_i'' - \omega_i, t(0)) > x_{i0}' + p^* \cdot x_i' + Q(x_i' - \omega_i, t(0))$. If $\bar{x}_i'' \notin V_{ik}$ then it would be unaffordable under $B(p^*, t(\gamma_i); \bar{\omega}_i)$, due to the $\gamma_i \cdot t_k^*$ queue cost. If $\bar{x}_i'' \in V_{ik}$, then it would also be affordable under $B(p^*, t^*; \bar{\omega}_i)$, yet would be strictly preferred to $\bar{x}_i^*$, contradicting $\bar{x}_i^* \in \xi_i(p^*, t^*)$. Thus, $\bar{x}_i' \in \xi_i(p^*, t(\gamma_i))$.

Next, we show that $\bar{x}_i^* \in \xi_i(p^*, t(\beta))$ for all $\beta \geq \gamma_i$. We have already established that $\bar{x}_i^* \in B(p^*, t(\beta); \bar{\omega}_i)$. Since $\bar{x}_i' \sim_i \bar{x}_i^*$ and $\bar{x}_i' \in \xi_i(p^*, t(\gamma_i))$, it follows that $\bar{x}_i^* \in \xi_i(p^*, t(\gamma_i))$. When $\beta > \gamma_i$, note that $B(p^*, t(\beta); \bar{\omega}_i) \subset B(p^*, t(\gamma_i); \bar{\omega}_i)$; thus, if some $\bar{x}_i'' \in B(p^*, t(\beta); \bar{\omega}_i)$ existed where $\bar{x}_i'' \succ_i \bar{x}_i^*$, then it would also have been available in $B(p^*, t(\gamma_i); \bar{\omega}_i)$, which would contradict $\bar{x}_i^* \in \xi_i(p^*, t(\gamma_i))$.

As a result, $\bar{x}^*$ can be supported as an Essential QRE using reduced queuing times $t(\gamma_{\max})$, where $\gamma_{\max} = \max_{i=1}^N \gamma_i$. At least one agent will be precisely indifferent between $\bar{x}_i^*$ and some allocation that involves trade of good k, allowing a measure zero to queue. For the rest of the agents, $\bar{x}_i^*$ will still be part of their demand under these reduced queues.

D Proof of Theorem 4

Assume that the conditions listed do not hold. Since $\bar{x}^*$ cannot be supported as a WE under any price vector, including p^*, it must be that $t^* \neq 0$. Without loss of generality, assume that $t_j > 0$.

Consider an agent i who makes a net purchase in market j, as well as one i' who makes a net sale. Define e_j as a unit vector with 0 everywhere except the j^{th} position. By strict monotonicity, $\bar{x}_i^* + \epsilon\, e_j \succ_i \bar{x}_i^*$ for all $\epsilon > 0$.

By continuity of preferences, there exists $\delta(\epsilon) > 0$ such that $\bar{x}_i^* + \epsilon\, e_j - \delta(\epsilon)\, e_0 \sim_i \bar{x}_i^*$. Moreover, since at least one such buyer does not have a kinked indifference curve, it must be that $\frac{\delta(\epsilon)}{\epsilon} \nearrow p_j^* + t_j^*$ as $\epsilon \to 0$; otherwise, it would violate the fact that $\bar{x}_i^* \in \xi_i(p^*, t^*)$. Similarly, there exists a $\gamma(\epsilon) > 0$ such that $\bar{x}_{i'}^* - \epsilon\, e_j + \gamma(\epsilon)\, e_0 \sim_{i'} \bar{x}_{i'}^*$, and $\frac{\gamma(\epsilon)}{\epsilon} \searrow p_j^*$ as $\epsilon \to 0$.

For ϵ sufficiently small, an α may be chosen such that $\gamma(\epsilon) < \alpha < \delta(\epsilon)$. By assigning agent i the bundle $\bar{x}_i^* + \epsilon\, e_j - \alpha\, e_0$ and agent i' the bundle $\bar{x}_{i'}^* - \epsilon\, e_j + \alpha\, e_0$, both are strictly better off, while all others are unharmed. Most importantly, constrained feasibility still holds; this was merely a transfer between two agents. Thus $\bar{x}^*$ is not constrained Pareto efficient.

If there are no buyers or sellers of good j, yet $t_j > 0$, the same procedure can be followed, just identifying the potential buyers by observing who would buy if $t = 0$. (Such a buyer or seller must exist in some market j that has $t_j \neq 0$; if not, then setting $t = 0$ would still be an equilibrium, which makes $\bar{x}^*$ supported as a WE.)

E Proof of Theorem 5

We start by proving the following claim (via three cases): For all i, $\bar{x}_i^* \succsim_i \hat{x}_i$ for all $\hat{x}_i \in \mathbb{R}_+^{\ell+1}$ where $\hat{x}_{i0} + p^*\hat{x}_i \leq \bar{x}_{i0} + p^*\bar{x}_i$.

First, consider $\hat{x}_i$ where $Q(\hat{x}_i - \omega_i, t^*) \leq Q(\bar{x}_i^* - \omega_i, t^*)$, which would occur if there is no queue in a market, or when $\hat{x}_i$ has i participating on the same side of each market as in $\bar{x}_i^*$, or switches from the side which has to queue to the side that doesn't. Since the money price of $\hat{x}_i$ is also weakly less than that of $\bar{x}_i^*$, then $\hat{x}_i$ must be affordable under $B(p^*, t^*; \bar{\omega}_i)$. Since $\bar{x}_i^* \in \xi_i(p^*, t^*)$, $\bar{x}_i^* \succsim_i \hat{x}_i$.

Next, suppose that $\hat{x}_i$ causes $Q(\hat{x}_i - \omega_i, t^*) > Q(\bar{x}_i^* - \omega_i, t^*)$ — in at least one market j, $\hat{x}_i$ has i switch from the side of the market that doesn't queue into the side that does. Call J the set of all markets in which this switch occurs. Suppose that $\hat{x}_i \succ_i \bar{x}_i^*$. By convexity of preferences, for any $\alpha \in (0, 1)$, $x_i^\alpha \equiv \alpha\hat{x}_i + (1 - \alpha)\bar{x}_i^* \succ_i \bar{x}_i^*$. Suppose that $\bar{x}_{ij}^* \neq \omega_{ij}$ for all $j \in J$. For large enough α, $Q(x_i^\alpha - \omega_i, t^*) = Q(\bar{x}_i^* - \omega_i, t^*)$, so $x_i^\alpha \in B(p^*, t^*; \bar{\omega}_i)$. But this contradicts $\bar{x}_i^* \in \xi_i(p^*, t^*)$.

If instead $\bar{x}_{ij}^* = \omega_{ij}$ for some $j \in J$, then by assumption, the same bundle is chosen by i if t_j is set to zero. Thus, we can exclude t_j from the queue time and return to whichever of the preceding two cases is appropriate.

The claim (now established) essentially states that $\bar{x}^*$ is Walrasian equilibrium (in the standard sense) of an economy with endowments $\bar{x}^*$ under equilibrium prices p^*. By applying the first fundamental welfare theorem in this economy, we are assured that $\bar{x}^*$ is Pareto optimal compared to any other allocation in $\mathbb{F}_C(\bar{x}^*)$.

F Proof of Proposition 2

Claim 1:

Let i be the net buyer of good 1 in the *f-QRE*. Note that net sellers are indifferent between the two QREs, since they are not queueing and face the same prices in both cases. Also, if $t^f = 0$ or $t^\rho = 0$ then this is a Walrasian Equilibrium, and thus there is a ρ-*QRE* and a *f-QRE* with the same prices and allocation. Welfare is equivalent in all three.

In both equilibria, net supply from type $-i$ is the same. Thus, $x_{i1}^f - \omega_{i1} = x_{i1}^\rho - \omega_{i1}$.

Of course, it is possible that $\bar{\xi}_i^f(p^*, t^f)$ is not single-valued. If so, then the allocation $\bar{x}_i^f$ is the convex combination of the endowment and some other bundle, with some fraction of type i agents choosing each, though indifferent between them. Thus, in the *f-QRE*, agents of type i will receive the same utility as in autarky. The endowment is always available in a ρ-*QRE*, so type i is at least as well off under proportional queues (and possibly better off).

If $x_{i1}^f = \omega_{i1}$, then no trade occurs in either equilibrium. Since no one trades, no one incurs the queuing cost, regardless of t^f and t^ρ. Thus each type of agent receives the same utility (namely, the autarky level) in one QRE as the other.

Finally, suppose $\bar{x}_i^f = \bar{\xi}_i^f(p^*, t^f)$ is single valued. Let $\bar{x}_i(p, \tilde{x}_i)$ solve the problem: $\min_{\bar{x}_i} x_{i0} + px_{i1}$ s.t. $\bar{x}_i \succcurlyeq_i \tilde{x}_i$.

Let $p \in (p^C, p^C + t^\rho]$. By assumption, $\xi_{i1}^f(p, 0) \geq \xi_{i1}^f(p^C + t^\rho, 0) = x_{i1}^\rho$. Moreover, if $\tilde{x}_{i1} = \bar{\xi}_i^f(p, 0)$, then by the law of compensated demand, $x_{i1}(p^C, \tilde{x}_i) > \tilde{x}_{i1}$.

Define $t(p) \equiv (1, p^C) \boldsymbol{\cdot} \left(\bar{\omega}_i - \bar{x}_i\left(p^C, \tilde{x}_{i1}\right)\right)$, which is the difference between endowed wealth under p^C and the wealth for which $\bar{x}_i\left(p^C, \tilde{x}_{i1}\right)$ will be the expenditure-minimizing choice. But then $\xi_{i1}^f(p^C, t(p)) = x_{i1}(p^C, \xi_{i1}^f(p, 0))$. So $\xi_{i1}^f(p^C, t(p)) > x_{i1}^\rho$, for all $p \in (p^C, p^C + t^\rho]$, which is equivalent to saying $\xi_{i1}^f(p^C, t) > x_{i1}^\rho$, for all $t \in (0, t(p^C)]$.

So to obtain market clearing, $t^f > t(p^C)$. But $t(p^C)$ is the maximum reduction in wealth (or queue time) for which i will be at least as well off as in the ρ-*QRE* allocation. Thus, $x_i^\rho \succcurlyeq_i x_i^f$, with strict preference if $t^f > 0$ and $x_i^f \succ_i \omega_i^f$.

Claim 2:

By assumption, net sellers will offer the same or less of good 1 when its price falls from p^w to p^C. The smaller budget set implies that sellers will be worse off. Since there is only one type of buyer (i), all of whom make the same net purchase, markets only clear if $x_{i1}^\rho \leq x_{i1}^w$. Yet by assumption, ξ_{i1} will increase if the effective price decreases. Thus, the effective price $p^C + t^\rho \geq p^w$ in order for the market to clear. A smaller budget set combined with monotonicity imply that $x_i^w \succcurlyeq_i x_i^\rho$. Indeed, the *WE* is strictly preferred to the ρ-*QRE* if the assumption holds with strict inequality for either buyer or seller.

Claim 3:

Follows directly from Claim 1 and 2.

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