

Searching on a Deadline

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Abstract

We analyze an equilibrium search model where the buyer seeks to purchase a good before a deadline. The buyer's reservation price rises continuously as the deadline approaches. A seller cannot observe a potential buyer's remaining time until deadline, and hence posts a price that weighs the probability of sale versus the profit once sold. The model has a unique equilibrium, which can take exactly one of two forms. In a late equilibrium, buyers initially forgo any purchases, only accepting some offers as the deadline draws near. In an early equilibrium, buyers are willing to accept some offers even as they enter the market. Equilibrium price dynamics are determined by the concentration of buyers near their deadline, as well as their urgency of completing the transaction before their deadline.

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1 Introduction

Quite frequently, the timing of a purchase is of crucial importance to the buyer. When a worker relocates to a new city, he would ideally like to secure a new home before the start date; otherwise, he may have to use costly temporary housing, such as hotels or short-term leases. Similarly, a parent might buy a birthday gift for her child at any time before the birthday, but purchasing it thereafter would cause significant grief. For cable television, cellular phones, credit cards and even home mortgages, consumers are often given an introductory rate that eventually expires; this may give the consumer incentive to search for a replacement provider as this deadline approaches.

These scenarios are non-trivial because finding the right home, gift, or provider requires search. If the right item at the best price were perfectly known, one could acquire it just before the deadline without worry. More likely, though, one will only come across an acceptable item infrequently. Moreover, that item could be offered at a variety of prices; indeed, the highest prices might only become acceptable as the deadline looms near.

Deadlines have been frequently studied in the bargaining literature. There, a buyer and a seller must agree on a price, effectively dividing their surplus (which may or may not be common knowledge). If they fail to reach an agreement before a commonly-known deadline, the available surplus is reduced. Here, we recast this deadline problem with the anonymity that is typical in a market — sellers are aware that buyers face a deadline, but are unsure how close any particular buyer is to that deadline.

In particular, buyers randomly encounter sellers of a homogeneous good. On meeting, the buyer learns the seller's asking price and must decide whether to make the purchase, or decline the opportunity and continue searching in the market. Buyers initially have a *grace period* to buy the good during which they enjoy a flow of high utility; after passing the deadline, however, their flow of utility drops until the purchase is completed. Thus, even though buyers identically value the good in question, they will be ex-post heterogeneous in their willingness to pay for it, depending on how close they are to the deadline.

Sellers explicitly consider this in choosing their asking price. A higher price will generate greater profit if accepted, but also limits the pool of buyers who would

accept the offer. In equilibrium, these two effects exactly cancel each other, allowing identical sellers to ask different prices and yet have equal expected profit.¹ We study this endogenous price formation in our model of search on a deadline.

The resulting equilibrium displays several interesting characteristics. First, the equilibrium is unique and takes one of two forms. In a late equilibrium, buyers initially forgo any purchases, preferring to wait until closer to the deadline. In an early equilibrium, buyers will accept some of the offered prices even as they enter the market. In both cases, their reservation price continuously rises (at an ever-increasing rate) as the clock ticks down, allowing more purchases to occur as a cohort of buyers approaches their deadline.

Second, our model generates a continuum of offered prices. In addition, the price distribution includes two atoms. One is at the highest price, which will only be accepted by the unlucky buyers who are late in their grace period. The other is at the lowest price, targeting those who have recently entered the market.

Third, price dynamics (analyzed through comparative statics) are largely determined by two factors: deadline concentration and deadline urgency. Concentration refers to the portion of the steady-state population of buyers that are near their deadline. Greater concentration encourages sellers to target these desperate buyers more heavily. Urgency refers to the utility drop after the deadline. A larger drop causes a steeper increase in reservation prices, as buyers are willing to pay more to avoid this painful utility reduction.

The interaction of these factors can have surprising consequences in equilibrium. For instance, if the good becomes more valuable to buyers (with no change in its cost of production), their eagerness to quickly obtain it will reduce deadline concentration. As a consequence, prices actually fall. If instead the seller's cost of production increases, the resulting price increase discourages early purchases. But this increases deadline concentration, allowing sellers to earn more profit after the cost increase.

If buyers enjoy higher grace period utility, they delay their purchases even longer to enjoy more of this utility flow. This increases deadline concentration, but it also increases deadline urgency. The change in utility at the deadline becomes more stark, causing reservation prices to increase more rapidly as time runs out. Both effects lead

¹This balanced tradeoff between markup and volume of sales is common to all models of price posting that generate dispersed prices, such as Rob (1985), Diamond (1987), Burdett and Mortensen (1998), etc. The unique approach of our model is that the impending deadline endogenously determines the population willing to accept a given price.

to higher prices.

Increasing the length of the grace period has no effect on the late equilibrium (neither concentration nor urgency are affected). Buyers merely delay their purchases until the same amount of time remains, and prices are unaltered. In the early equilibrium, however, a longer grace period encourages slower acceptance and thus greater deadline concentration. Ironically, this allows for higher prices in equilibrium.

The technical challenge of this model is that the distribution of seller's asking prices is endogenously determined. We overcome this by translating equilibrium conditions into a differential equation which is analytically solvable. van den Berg (1990) implemented this approach for unemployment search with an exogenous wage distribution. Its first application with an endogenous wage distribution appears in Akin and Platt (2011). There, unemployed workers receive unemployment insurance benefits for a finite duration, after which they are cut off. This creates ex-post differences among the workers' reservation wages, which allows firms to offer these otherwise identical workers different wages.

We proceed as follows: Section 2 presents the baseline model and defines equilibrium. In Section 3, we walk through the process of translating the equilibrium conditions, and present the equilibrium solution. Section 4 summarizes and provides intuition for comparative statics. Section 5 extends the model for more general applications, by allowing for delayed consumption of the new good and early termination fees. We defer full review of related work until Section 6, which compares our results to those in the search and bargaining literatures. We conclude in Section 7.

2 Model

Consider a continuous time environment, with infinitesimal buyers and sellers each entering the market at rate δ . All agents discount future utility at rate ρ . The good being sold is homogeneous, which provides a present value of $\frac{x}{\rho}$ to any buyer, but because sellers may ask different prices for the good, buyers may find it worthwhile to search. Buyers and sellers encounter one another at Poisson rate λ ; at that point, the buyer draws an asking price p from the distribution of offered prices, $F(p)$. The buyer can either make the purchase, obtaining $\frac{x}{\rho} - p$ surplus, or continue searching (with no recall of past offers).

We assume throughout that $e^{-\frac{\rho T}{2}} > \frac{\rho}{\lambda}$. This ensures that buyers encounter a seller

with sufficient frequency that continued search is a viable option. If this did not hold, buyers would always accept the first offer they encounter, and all sellers would charge the same (monopoly) price.

2.1 The Buyer's Search Problem

The buyer has T units of time to search without penalty, which we refer to as the *grace period*. During this time, he receives utility b each instant. After the grace period expires, the instantaneous utility falls to $d < b$ until a purchase is made.² We characterize this search problem using the remaining time until the grace period expires, s , as the state, and endogenously determine the buyer's reservation price $R(s)$ for each state.

For instance, once the grace period expires ($s = 0$), the buyer's problem can be recursively formulated as follows:

$$\rho V(0) = \max_{R(0)} d + \lambda \int_{-\infty}^{R(0)} \left(\frac{x}{\rho} - p - V(0) \right) dF(p). \quad (1)$$

Here, $V(0)$ represents the expected net present utility of a buyer after the grace period. Each instant, he receives utility d . He also encounters purchase opportunities at rate λ , and will accept any price at or below $R(0)$. If a transaction occurs at price p , his utility changes from $V(0)$ to $\frac{x}{\rho} - p$.

During the grace period, the recursive problem takes the following form:

$$\rho V(s) = \max_{R(s)} b - V'(s) + \lambda \int_{-\infty}^{R(s)} \left(\frac{x}{\rho} - p - V(s) \right) dF(p). \quad (2)$$

Note three changes in this Bellman equation, compared to decisions after the grace period. First, the instantaneous utility is b . Second, a buyer with grace time remaining could hold out for a lower price $R(s)$. Finally, the state variable s deterministically falls as the grace period ticks down, which is reflected in the term $-V'(s)$.

By defining the Bellman equation in this way, we are assuming that both $V(s)$ and

²In housing search, b would reflect the buyer's consumer surplus in his current housing, while d reflects the lower surplus of switching to short-term housing or long-distance commuting. In gift search, b would be the stream of utility from a relationship (in which case it might be natural to set $x = b$, so that the gift maintains the status quo of the relationship), while d reflects the lower utility when expectations of a gift are not met.

$V'(s)$ are continuous and differentiable; thus we do not examine possible equilibria with discontinuous value functions. Even though the instantaneous utility abruptly falls after once $s = 0$, the present expected cost of these penalties grow in a smooth way as the grace period nears expiration.

In formulating a reservation price, the buyer should make a purchase as long as it weakly increases his utility. Thus, for all $s \in [0, T]$:

$$R(s) = \frac{x}{\rho} - V(s). \quad (3)$$

2.2 Steady State Conditions

For sellers to choose a pricing strategy, it will be critical to know how many buyers there are at each state of the search process. We consider a steady state equilibrium, where the measure of buyers in each state stays constant over time. Let $H(s)$ denote the measure of buyers with s or less time remaining in their grace period. Note that $H(0)$ includes all whose grace period has expired. Then $H'(s)$ indicates the relative density of buyers in state s .

Buyers enter the market at rate δ ; thus,

$$H'(s) = \delta. \quad (4)$$

At state $s > 0$ in the grace period, buyers exit the market only when they find an acceptable price, which happens at rate $\lambda F(R(s))$. Thus, the density of buyers at s must fall at that rate:

$$H''(s) = \lambda F(R(s))H'(s). \quad (5)$$

Finally, among those who have exceeded the grace period ($s = 0$), all prices offered in equilibrium are acceptable. Thus, they exit whenever they encounter a seller, *i.e.* at rate λ . At the same time, this population of expired buyers is replenished by the flow of buyers whose grace period has just expired, $H'(0)$:

$$H'(0) = \lambda H(0). \quad (6)$$

2.3 The Seller's Problem

Sellers produce their good at cost $c < x$, at the time of the transaction. They are unable to observe the state of the buyer with whom they have been paired. Thus, asking a higher price bears the risk of a lower likelihood of being accepted. At the same time, it would result in higher realized profits if accepted. This is represented as follows:

$$\pi = \lambda \frac{H(R^{-1}(p))}{H(T)}(p - c). \quad (7)$$

Here, $R^{-1}(p)$ is the inverse of the reservation price function, so $H(R^{-1}(p))$ denotes the measure of buyers willing to accept such a price, while $H(T)$ is the total population of active buyers.

If multiple prices produce the same maximal expected profit, sellers can randomize over these prices, which would be represented in the cumulative price distribution $F(p)$. One can interpret this as each seller using the same mixed strategy, randomizing anew for each potential buyer. Alternatively, each seller could stick with a particular price, with $F(p)$ representing the aggregate distribution of sellers' choices. Since there is no repeated interaction between any given buyer or seller, either interpretation is equally valid.

2.4 Equilibrium Definition

A steady state search equilibrium consists of seller profit π , a reservation price function $R(s)$, the measure of buyers $H(s)$, and the distribution of sellers' offered prices $F(p)$, such that:

1. $R(s)$ maximizes utility of a buyer with s time until the expiration of the grace period, given $F(p)$.
2. All prices in the support of F produce the same profit π , while all other prices produce no more than π .
3. $H(s)$ satisfies the steady state conditions in Eqs. 4 through 6.

3 Equilibrium Characterization

We now demonstrate the process of solving for equilibrium. This begins by translating our Bellman equations and steady state conditions into a second-order differential equation of the reservation prices, $R(s)$. That is, we will find a smooth reservation price function that obeys all the necessary conditions for equilibrium. Indeed, among such smooth functions, this solution will be unique. Having solved this, we can derive all the other equilibrium objects.

In characterizing the equilibrium, it is useful to do so in terms of a critical state $S^* \in [0, T]$ that will be determined by our equilibrium conditions. Any buyer with more than S^* time until expiration will have reservation prices that are below anything offered in equilibrium; that is, they are content to continue enjoying their grace-period utility stream rather than make a purchase. Buyers with less than S^* time remaining will be willing to accept at least some of the offered prices. We refer to this as a *late equilibrium*, because buyers wait until later in their grace period before making purchases.

It is also possible to obtain an *early equilibrium*, in which buyers will accept at least some of the available prices at the beginning of their grace period. This type of equilibrium also depends on an endogenous critical state $Z^* \in [0, T]$. Sellers do not find it profitable to post prices that would only be accepted by buyers with less than Z^* time remaining, because there are too few of these buyers in the market.

For expositional clarity, we only solve for the late equilibria in this section. The early equilibria are derived in a similar manner, presented in Appendix A.2.

3.1 Translating the Bellman Equations

Since $V(s)$ and $V'(s)$ are continuous and differentiable, we immediately conclude from Equation 3 that $R(s)$ and $R'(s)$ must be as well. In particular, $R'(s) = -V'(s)$ and $R''(s) = -V''(s)$. Next, we take the derivative of Equation 2 with respect to s , which eliminates the integral:

$$(\rho + \lambda F(R(s))) V'(s) + V''(s) = \lambda \left(\frac{x}{\rho} - R(s) - V(s) \right) F'(R(s)) R'(s).$$

But since $R(s) = \frac{x}{\rho} - V(s)$, the last term drops out. We then substitute for $V'(s)$ and $V''(s)$, expressing the equation above entirely in terms of the reservation price

and price distribution:

$$(\rho + \lambda F(R(s))) R'(s) = -R''(s). \quad (8)$$

Next, we must ensure that $V(s)$ is continuous at $s = 0$. Comparing the definitions in Eqs. 1 and 2, we see that continuity will only hold if $V'(0) = b - d$, or in terms of the reservation price:

$$R'(0) = d - b. \quad (9)$$

Similarly, we must ensure that continuity also holds at $s = S^*$. For $s > S^*$, the reservation price is below any price offered in equilibrium; thus, in this range, $\rho V(s) = b - V'(s)$. Substituting the reservation price function in place of the value function, at $s = S^*$ we require:

$$x - b - \rho R(S^*) = R'(S^*). \quad (10)$$

3.2 Translating the Steady State Conditions

Here, we begin with the equal profit condition in Equation 7. The seller must earn the same expected profit whether offering price $R(S^*)$ or some higher price $R(s)$:

$$\frac{H(S^*)}{H(T)}(R(S^*) - c) = \frac{H(s)}{H(T)}(R(s) - c). \quad (11)$$

Taking the first and second derivatives of this condition, we find:

$$\begin{aligned} H'(s) &= -\frac{H(S^*)(R(S^*) - c)R'(s)}{(R(s) - c)^2} \text{ and} \\ H''(s) &= \frac{H(S^*)(R(S^*) - c)(2R'(s)^2 - (R(s) - c)R''(s))}{(R(s) - c)^3}. \end{aligned}$$

We then substitute these into the steady state equations, obtaining:

$$R'(0) = \lambda(c - R(0)), \quad (12)$$

$$F(R(s)) = \frac{R''(s) - \frac{2R'(s)^2}{R(s) - c}}{\lambda R'(s)}, \text{ and} \quad (13)$$

$$R'(S^*) = -\frac{\delta(R(S^*) - c)}{H(S^*)}. \quad (14)$$

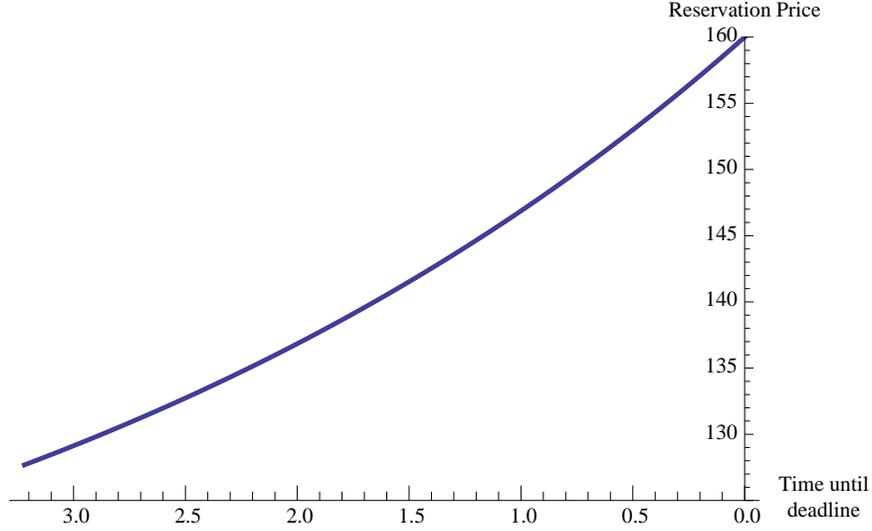


Figure 1: Buyers' reservation prices $R(s)$ as a function of remaining grace period time.

3.3 Equilibrium Solution

The final step is to solve this system of differential equations. In particular, by substituting for $F(R(s))$ in Eq. 8 using Eq. 13, we obtain:

$$R'(s) \left(\rho + \frac{2R'(s)}{(c - R(s))} \right) + 2R''(s) = 0. \quad (15)$$

We solve this second-order differential equation together with Eqs. 9 and 12 as boundary conditions, and obtain the unique solution:

$$R(s) = c + \frac{b - d}{\lambda} e^{-\frac{2\lambda}{\rho}(1 - e^{-\frac{\rho s}{2}})} \text{ for } s \in [0, S^*]. \quad (16)$$

Notice that $R(s)$ is strictly decreasing in s , indicating that buyers are willing to accept higher prices as their remaining grace period dwindles. Moreover, $R''(s) > 0$, which is to say the price increase becomes more pronounced as the deadline approaches. These dynamics are illustrated in Figure 1.³

This still leaves us with two other boundary conditions at S^* , Eqs. 10 and 14. The latter simply determines $H(S^*)$ and hence the total population of active buyers, $H(T)$,

³Parameters used were: $T = 12$, $b = x = 5$, $c = 100$, $d = -10$, $\lambda = 0.25$, and $\rho = 0.05$. Under these parameters, $S^* = 3.22$.

which is presented below. The former implicitly determines S^* . By substituting for $R(S^*)$ and $R'(S^*)$ found in Eq. 16 into Eq. 10, we get the following equation:

$$\phi(S) \equiv \frac{x-b}{\rho} - c + \frac{b-d}{\rho} \left(e^{-\frac{\rho S}{2}} - \frac{\rho}{\lambda} \right) e^{-\frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho S}{2}} \right)}. \quad (17)$$

This equation compares the net surplus of selling this good, $\frac{x-b}{\rho} - c$, to the expected net present loss from exceeding the deadline, $b-d$. In doing so, it accounts for the expected search duration, including that the deadline may not be reached at all. This is computed taking as given that buyers only accept (some) prices once they have S time or less remaining of their grace period. To ensure that the Bellman Equation (Eq. 2) is continuous at S^* , we need $\phi(S^*) = 0$.

The remaining equilibrium objects can be computed using this solution for $R(s)$. The distribution of seller asking prices would be:

$$F(p) = \begin{cases} 0 & \text{if } p < R(S^*) \\ 1 - \frac{\rho}{2\lambda} \left(1 - \ln \frac{\lambda(p-c)}{b-d} \right) & \text{if } R(S^*) < p < c + \frac{b-d}{\lambda} \\ 1 & \text{if } p \geq c + \frac{b-d}{\lambda}. \end{cases} \quad (18)$$

Alternatively, this can be expressed in terms of which buyers are being targeted by a particular seller:

$$F(R(s)) = \begin{cases} 0 & \text{if } s > S^* \\ e^{-\frac{\rho s}{2}} - \frac{\rho}{2\lambda} & \text{if } 0 < s < S^* \\ 1 & \text{if } s = 0. \end{cases}$$

Note that there are atoms of sellers offering the lowest price and the highest price. Figure 2 plots the $F(p)$ function.

The population of buyers can also be quickly derived from the equal profit condition in Eq. 11:

$$H(s) = \begin{cases} \frac{\delta}{\lambda} e^{\frac{\rho S^*}{2} - \frac{2\lambda}{\rho} \left(e^{-\frac{\rho s}{2}} - e^{-\frac{\rho S^*}{2}} \right)} & \text{if } 0 \leq s \leq S^* \\ \delta(s - S^*) + \frac{\delta}{\lambda} e^{\frac{\rho S^*}{2}} & \text{if } S^* < s \leq T, \end{cases} \quad (19)$$

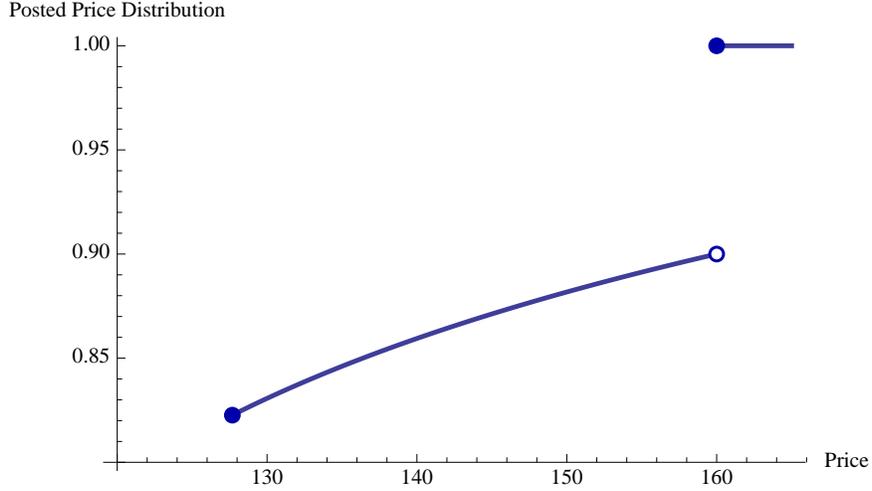


Figure 2: Cumulative distribution $F(p)$ of prices offered by sellers, with $F(p) = 0$ for all $p < 127.7$.

and sellers earn expected profit:

$$\pi = \frac{b - d}{\lambda(T - S^*) + e^{\frac{\rho S^*}{2}}} e^{\frac{\rho S^*}{2} - \frac{2\lambda}{\rho}} \left(1 - e^{-\frac{\rho S^*}{2}}\right) \text{ if } p \in [R(S^*), R(0)]. \quad (20)$$

Note that profits do not depend on the price p , because choosing a higher p entails a lower likelihood of acceptance. Figure 3 illustrates this by computing, for a seller who posts a given price, the average number of customers he will encounter over the average number of sales he will complete. The lowest price is accepted by any customer that encounters the seller, but a seller charging the highest price will make more than two offers for each acceptance.

While π indicates that any price in the support of $F(p)$ is equally profitable, we must verify that any price outside the support is no more profitable. Indeed, this was the only equilibrium requirement that was not used (and hence will not be automatically satisfied) in the construction of this proposed equilibrium. This verification is a relatively simple task. Any price above $R(0)$ is rejected by everyone, and thus earns zero profit. A price below $R(S^*)$, on the other hand, will result in a few additional purchases (from those with $s > S^*$), but not enough to compensate for the lower markup. This latter argument is demonstrated in the proof of the following proposition.

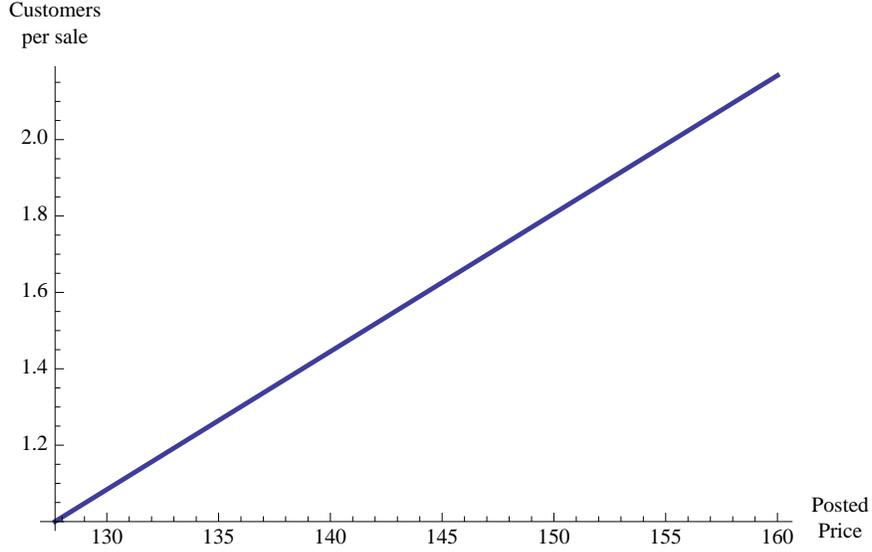


Figure 3: For a seller who posts a given price p , the average number of customers he encounters per sale he completes.

Proposition 1. *Assuming there exists $Z^* \in [0, T]$ such that $\phi(Z^*) = 0$, Eqs. 16, 18, 19, and 20 constitute an equilibrium.*

To this point, we have proceeded as if the solution to $\phi(S^*) = 0$ yields an $S^* \in [0, T]$. If $\phi(S) < 0$ for all $S \in [0, T]$, then a *late degenerate equilibrium* emerges in which all sellers offer the same price $p = \frac{x-d}{\rho}$, and all buyers reject this price until their grace period has expired.⁴ Intuitively, this would only occur if the benefits from the purchase are relatively small compared to the penalty from exhausting the grace period, making the buyer willing to wait out the full grace period but purchase at the first opportunity thereafter. Indeed, we show in Proposition 2 that this only occurs if and only if $\phi(0) \leq 0$, which is equivalent to $\frac{b-d}{\lambda} \geq \frac{x-d}{\rho} - c$.

On the other hand, if $\phi(S) > 0$ for all $S \in [0, T]$, buyers are so anxious to purchase that they are willing to accept at least some offers immediately on entering the market. Under such circumstances, no late equilibria exists, but an early equilibrium will exist, which we solve for in Appendix A.2. There, we define $\psi(Z)$ in Eq. 44, which plays a role analogous to $\phi(S)$ in Eq. 17.

Significantly, these two equations coincide at $\psi(0) = \phi(T)$, which is the dividing

⁴Note that when $\phi(0) = 0$, the only price offered is $R(0) = c + \frac{b-d}{\lambda}$, which is the same as $p = \frac{x-d}{\rho}$ after substitution using $\phi(0) = 0$.

line between early and late equilibria. Only late equilibria will exist if $\phi(T) \leq 0$. This condition is equivalent to:

$$\frac{x - b}{\rho} + \frac{b - d}{\rho} \left(e^{-\frac{\rho T}{2}} - \frac{\rho}{\lambda} \right) e^{-\frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho T}{2}} \right)} \leq c.$$

Indeed, if one sets $x = b$, this simply says that the cost of producing the new good must be greater than the expected harm from delayed acceptance of offers. As long as the grace period T is reasonably long, the arrival rate of offers λ is somewhat frequent, or the penalty for expiration $b - d$ is not too large, this will hold. If reversed, however, then only early equilibria will exist.

We can be more precise if we slightly strengthen our assumption that compares the offer arrival rate to the discount rate. If $e^{-\frac{\rho T}{2}} > \frac{2\rho}{\lambda}$, then whichever equilibrium occurs will be unique; no other smooth solution exists.

Proposition 2. *Assuming $e^{-\frac{\rho T}{2}} > \frac{2\rho}{\lambda}$, exactly one of the following will occur:*

- *the late degenerate equilibrium, when $\phi(0) \leq 0$.*
- *a unique late dispersed equilibrium, when $\phi(0) > 0 \geq \phi(T)$.*
- *a unique early dispersed equilibrium, when $\psi(0) > 0 > \psi(T)$.*
- *the early degenerate equilibrium, when $\psi(T) \geq 0$.*

Perhaps it is surprising that the degenerate equilibrium does not always exist, as is often the case in search models. Usually, this occurs because if buyers expect a single price to be offered, they should accept that price whenever it is encountered. Here that is not true; some buyers will prefer to enjoy their remaining grace period utility rather than accept a high price. When $\phi(0) > 0$, there are enough of these buyers to make it profitable for some sellers to target these buyers by offering a lower price. Thus the degenerate equilibrium cannot be sustained.

We primarily focus our analysis on the late dispersed equilibrium. This is partly for brevity, and we find this to be the more compelling case for most of our motivating examples. Few people start shopping for next year's birthday gift immediately after this year's party, nor start hunting for a replacement cell phone contract immediately on signing their current contract. This is because utility during the grace period is sufficiently high and offers are sufficiently frequent to justify ignoring all offers for a time.

4 Comparative Statics

Perhaps the best way to understand the importance of the equilibrium search process is to examine comparative statics on several key parameters. In particular, the price response reveals how sellers react to these changes, and the consequences are often counterintuitive.

The following comparative statics are computed for both early and late dispersed equilibria;⁵ for the former, we assume that $e^{-\frac{\rho T}{2}} > \frac{2\rho}{\lambda}$. Of course, equilibrium is specified in terms of S^* or Z^* , which are implicitly solved from $\phi(S^*) = 0$ or $\psi(Z^*) = 0$. Yet implicit differentiation yields unambiguous signs on how S^* and Z^* respond to changes in parameters.

Lemma 1. *In a late dispersed equilibrium, $\frac{\partial S^*}{\partial d} < 0$, $\frac{\partial S^*}{\partial x} > 0$, $\frac{\partial S^*}{\partial c} < 0$, $\frac{\partial S^*}{\partial b} < 0$, and $\frac{\partial S^*}{\partial T} = 0$. Likewise, $\frac{\partial Z^*}{\partial d} < 0$, $\frac{\partial Z^*}{\partial x} > 0$, $\frac{\partial Z^*}{\partial c} < 0$, $\frac{\partial Z^*}{\partial b} < 0$, and $\frac{\partial Z^*}{\partial T} < 0$ in an early dispersed equilibrium.*

Table 1: Comparative Statics for Late (Early) Dispersed Equilibria

Statistic	$\partial/\partial b$	$\partial/\partial c$	$\partial/\partial x$	$\partial/\partial d$	$\partial/\partial T$
S^* (Z^*)	–	–	+	–	0 (–)
p_{\max}	+	+	0 (–)	– (+)	0 (+)
p_{\min}	+	+	–	– (+)	0 (+)
$H(0)$	+	+	–	+	0 (+)
$H(T)$	+	+	–	+	+
π	+	+	–	– (+)	– (+)

Notes: Signs listed in parenthesis apply only to the early dispersed equilibrium. In cells with no parenthesis, the sign is the same for early and late equilibria.

The comparative statics on prices can largely be explained in terms of two deadline effects: *concentration* and *urgency*. For the former, if buyers are concentrated more heavily near their deadline, sellers will offer with greater frequency the higher prices that target these late buyers. For the latter, the trajectory of buyers' reservation prices are greatly influenced by the gap between pre- and post-grace-period benefits. As $b - d$ increases, the reservation price increases more dramatically just

⁵Comparative statics for the early and the late degenerate equilibria are trivial to compute.

before the deadline. The buyer feels greater urgency to complete the purchase and avoid expiration, and is thus willing to pay more.

For interpretation of the comparative statics, we can consider the model as applied to housing search. In particular, suppose a household is receiving utility b from its current housing, and will receive utility x in its new housing (perhaps in another city) once it is obtained. If the search exceeds the deadline, however, the household's utility drops to d , as short-term housing or long-distance commuting is required until the new permanent housing is obtained.

4.1 An Increase in Grace-Period Utility, b

Suppose that the buyer enjoys greater utility during the grace period; *e.g.* the current housing (at the prior job location) is more enjoyable. Thus buyers want to reap more of the flow of current benefits rather than rush to the new purchase, inducing a delay in acceptance. This is reflected in the fact that S^* (or Z^*) decreases and $H(0)$ increases; buyers wait longer into the grace period to accept offers.

At the same time, this widens the gap between b and d . As the clock counts down, buyers become more anxious to secure a purchase and are willing to pay more to do so. Essentially, by enjoying more of their grace period, they have given themselves a shorter window for purchasing and hence a lower probability of doing so before expiration. Thus, greater concentration and greater urgency enable sellers to offer higher prices, targeting those who are closer to expiration.

Note that an increase in the production cost of the good, c , moves in the same direction as an increase in b . Indeed, what is surprising is that the sellers are actually more profitable after cost increase. Absent any change in S^* , prices would increase exactly enough to cover the cost increase. But this induces buyers to delay their purchase (so S^* falls); thus, sellers are more likely to encounter desperate buyers, and this increased concentration enables greater profit.

4.2 An Increase in Purchase Value, x

Suppose buyers experience greater utility from the new purchase; *e.g.* the permanent housing (at the new job location) is more desirable. In this baseline model, recall that on making the purchase, the buyer immediately takes possession of the good, switching utility x in place of b . Thus, a larger x makes the buyer want to transition

to this new location sooner. They find acceptable offers earlier in their grace period, and fewer of them reach expiration.

Since this reduces deadline concentration, the effect is almost the opposite of an increase in b . The only difference in sign occurs for the late equilibrium: the highest price $R(0)$ does not depend on x or S^* . (In the early equilibrium, $R(Z^*)$ is only reduced because the increase in x increases Z^* .) In magnitude, though, an increase in x typically has a smaller impact than an equal decrease in b , because there is no change in deadline urgency.

Ironically, then, sellers end up offering lower prices as customers value their product more. This highlights how the search friction is endogenously amplified or reduced by the buyers. When they are more eager to complete a purchase, they reject fewer offers; but this means fewer of them will exhaust their grace period, which reduces the market power of sellers.

4.3 An Increase in Post-Expiration Utility, d

Next suppose that the instantaneous utility after the grace period were to increase; *e.g.* short-term housing (at the new job location) is not as onerous. This makes buyers less concerned about reaching expiration, and thus they delay purchases in favor of additional search. However, this increase in concentration is mitigated by a decrease in urgency: the gap between b and d falls, making for a less dramatic change approaching the deadline.

In the early equilibrium, the concentration effect dominates. The increase in buyers near the deadline outweighs the flattening of their reservation price function, allowing sellers to charge and earn more. In the late equilibrium, however, this is reversed. In spite of the potential rush of buyers just before the deadline, the prices they are willing to pay fall disproportionately more. Intuitively, in the late equilibrium, purchases occur closer to the deadline, hence the $b - d$ gap has greater influence on the (relevant) reservation prices.

4.4 A Longer Grace Period, T

Finally, we ask what would happen if buyers could start their search earlier. Surprisingly, this has no effect on the equilibrium choices in the late equilibrium. With a little reflection, however, this is quite intuitive. Having extra time does not change their

urgency near the deadline (since only time remaining is relevant, not time spent), and buyers were not planning to make any purchases until S^* anyhow. Thus extra time is spent waiting (and enjoying b).

Indeed, the only tangible effect is that there will be more buyers in the market (near $H(T)$), but they might as well not be, since they accept no offers. The decrease in expected profit reflects the fact that now more of the buyers that sellers encounter will be in a state $s > S^*$ and thus refuse any offer.

In the early equilibrium, however, buyers are willing to accept at least some purchases even at the beginning of their search. One would thus expect that a longer grace period should help the buyers, allowing them to be more patient to find a good deal. Indeed, they do wait longer (Z^* falls and $H(0)$ rises), but this increases the deadline concentration. Sellers exploit this by raising prices and more heavily targeting buyers late in their grace period.

Of course, as T continues to rise, eventually Z^* falls 0, after which we shift to the late dispersed equilibrium. Beyond that point, buyers feel free to reject all offers as they first enter the market, and sellers can no longer exploit their longer grace period.

5 Extensions

Our model can easily be extended to include other plausible features of search on a deadline. In each case, the solution technique is the same, and the model's predictions are robust to these extensions. We only present the late dispersed equilibrium in each case; but one can derive the degenerate and early equilibria as in the baseline model, too.

5.1 Delayed Consumption

In the baseline model, on accepting an offer, the consumer immediately replaces utility flow b or d with the utility from x from the new purchase. In housing search, one might consider a household moving from Miami to Minneapolis. This assumption would mean that as soon as the new housing is secured, the household can immediately exit the old housing and begin enjoying the new.

Here, we consider when the buyer does not switch housing until the deadline has passed. For instance, the deadline might represent the start date of a new job in

Minneapolis, but the buyer must continue working in Miami up until then. If so, the household enjoys b up until the deadline regardless of when the purchase is made. If the deadline is crossed, however, the household must use expensive short-term living arrangements d until the permanent housing is secured.

This can be modeled simply by replacing the grace period Bellman equation (Eq. 2) with:

$$\rho V(s) = b - V'(s) + \lambda \int_{-\infty}^{R(s)} \left(\frac{b(1 - e^{-\rho s}) + x e^{-\rho s}}{\rho} - p - V(s) \right) dF(p). \quad (21)$$

Note that buyers who are past the deadline face the same problem as in Eq. 1, since they will immediately switch from the short-term arrangements to permanent housing as soon as it is secured. Also, the seller's problem is unchanged, as we assume that the good changes hands at the time of purchase, even if it sits idle until the grace period is over.

The solution in this environment is nearly identical to the baseline model. The key difference is that the grace period utility b has no effect on the equilibrium. This is not surprising, since the buyer receives the full flow of b for T periods regardless of purchase timing; it is entirely sunk. In words, the niceness of your current home has no effect on your purchase decision, since you will leave that home at the same time regardless. The specific solutions are:

$$R(s) = c + \frac{x - d}{\lambda} e^{-\frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho s}{2}}\right)} \text{ for } s \in [0, S^*], \quad (22)$$

with S^* determined by $\phi(S^*) = 0$, where

$$\phi(S) \equiv -\rho c - (x - d) \left(\frac{\rho}{\lambda} - e^{-\frac{\rho S}{2}} \right) e^{-\frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho S}{2}}\right)}. \quad (23)$$

The price distribution is:

$$F(p) = \begin{cases} 0 & \text{if } p < R(S^*) \\ 1 - \frac{\rho}{2\lambda} \left(1 - \ln \frac{\lambda(p-c)}{x-d} \right) & \text{if } R(S^*) < p < c + \frac{x-d}{\lambda} \\ 1 & \text{if } p \geq c + \frac{x-d}{\lambda}, \end{cases}$$

while $F(R(s))$ and $H(s)$ are identical to the baseline model.

5.2 Contracts and Termination Fees

A natural environment for search on a deadline comes from services that lock in a rate for a fixed period of time, such as cable television. New customers are offered an introductory rate in exchange for a two year commitment, for instance, after which the monthly price could be much higher. As the contract nears its conclusion, the customer becomes interested in shopping for a new contract. Of course, while the customer wants to avoid the rate increase after the contract expires, he is already committed to service with his current provider through the end of the contract, and would like to avoid early termination fees.

Here, we alter the model in three ways to reflect this environment. First, we assume that the population of consumers is fixed, all of whom currently have cable service. They are only distinguished by how much time (if any) they have remaining on their current contract. Thus, unlike our baseline model, the arrival rate δ of consumers in state $s = T$ is endogenous, determined by the rate at which old consumers sign new contracts.

Second, we assume that cable service always provides instantaneous utility b . When the consumer first signs a contract, they pay p upfront which covers all T periods of service; after the contract expires, they pay m each instant for continued service. Thus, d is replaced by $b - m$ in the post-expiration Bellman equation.

Third, we also introduce an early termination fee, z . If a consumer signs a new contract while s periods remain on his current contract, he must pay $(1 - e^{-\rho s})z/\rho$ dollars to the old provider, *i.e.* z dollars for each of the s remaining periods, in present value terms. If $z < 0$, this becomes a partial refund of the prepaid contract fee, p .

Here, we take m and z as exogenously given and constant across all firms, but the various service providers randomize in their initial contract price, p , which is endogenously determined.

These alterations are depicted by replacing the grace period Bellman equation (Eq. 2) with:

$$\rho V(s) = b - V'(s) + \lambda \int_{-\infty}^{R(s)} \left(V(T) - p - \frac{1 - e^{-\rho s}}{\rho} z - V(s) \right) dF(p), \quad (24)$$

and the post-expiration Bellman equation (Eq. 1) with:

$$\rho V(0) = b - m + \lambda \int_{-\infty}^{R(0)} (V(T) - p - V(0)) dF(p). \quad (25)$$

We also add one additional steady state requirement to denote the rate at which customers begin new contracts:

$$\delta = \lambda \left(H(0) + \int_0^{S^*} F(R(s)) H'(s) ds \right). \quad (26)$$

Finally, firms have additional sources of revenue from early termination fees and post-expiration service (as well as any change in the cost of provision from ending service early or extending it beyond T periods). The exact computation of these revenues and costs depend on the rate at which customers abandon their old contracts. Notice, though, that whatever the rate is, it does not depend on the contract price p . Since customers prepaid p for their old contract, it is a sunk cost and does not affect their decision of when to accept a new contract. Thus, we can let k denote the present value of the expected provision costs on a new contract, net of expected termination or post-expiration fees (which we explicitly calculate later). Then profits are given by:

$$\pi = \frac{H(R^{-1}(p))}{H(T)} (p - k). \quad (27)$$

Here, the solution is slightly more complicated than the baseline model, but has the same flavor. Again, utility b has no effect on the equilibrium. The specific solution is:

$$R(s) = k + \frac{m + z}{\lambda} e^{-\frac{2\lambda}{\rho} (1 - e^{-\frac{\rho s}{2}})} \text{ for } s \in [0, S^*], \quad (28)$$

with S^* determined by $\phi(S^*) = 0$, where

$$\phi(S) \equiv z (1 - e^{-\rho T}) + \rho k + (m + z) \left(\frac{\rho}{\lambda} - e^{-\frac{\rho S}{2}} + e^{\frac{\rho(S-2T)}{2}} \right) e^{-\frac{2\lambda}{\rho} (1 - e^{-\frac{\rho S}{2}})}. \quad (29)$$

The price distribution is:

$$F(p) = \begin{cases} 0 & \text{if } p < R(S^*) \\ 1 - \frac{\rho}{2\lambda} \left(1 - \ln \frac{\lambda(p-k)}{m+z} \right) & \text{if } R(S^*) < p < k + \frac{m+z}{\lambda} \\ 1 & \text{if } p \geq k + \frac{m+z}{\lambda}, \end{cases}$$

while $F(R(s))$ is identical to the baseline model. The distribution of consumers is only affected because δ is now endogenous; when solved, it yields:

$$H(s) = \begin{cases} \frac{e^{\frac{\rho S^*}{2} - \frac{2\lambda}{\rho} \left(e^{-\frac{\rho s}{2}} - e^{-\frac{\rho S^*}{2}} \right)}}{\lambda(T-S^*) + e^{\frac{\rho S^*}{2}}} & \text{if } 0 \leq s \leq S^* \\ \frac{\lambda(s-S^*) + e^{\frac{\rho S^*}{2}}}{\lambda(T-S^*) + e^{\frac{\rho S^*}{2}}} & \text{if } S^* < s \leq T. \end{cases}$$

The expected cost of a new customer solves to:

$$k = \frac{1 + e^{-\rho S^*}}{\rho} c + \frac{e^{-\rho S^*} + e^{-\rho T}}{\rho} z - e^{-\frac{2\lambda}{\rho} \left(1 - e^{\frac{1}{2}(S^*-T)\rho} \right) - \frac{1}{2}(S^*+T)\rho} * \left(\frac{2(c+z)}{\rho} + \frac{m-c}{\lambda+\rho} + \frac{4(c+z)\lambda}{\rho^2} \left(\int_{-\frac{2\lambda}{\rho}}^{-\frac{2e^{\frac{1}{2}(S^*-T)\rho\lambda}}{\rho}} \frac{e^{-t}}{t} dt \right) \right).$$

For a more succinct expression, consider the case in which the marginal cost of provision $c = 0$, as could well be the case with cable service. Furthermore, suppose that $z = 0$, so there is no loss in abandoning a contract early beyond the fact that the full term of the contract has been prepaid in p . In that case, the expected net cost of the customer is negative; that is, the cable company incurs no costs for a new customer and expects some extra revenue from those who do not find a new contract before the deadline:

$$k = -\frac{m}{\lambda+\rho} e^{-\frac{2\lambda}{\rho} \left(1 - e^{\frac{1}{2}(S^*-T)\rho} \right) - \frac{1}{2}(S^*+T)\rho}.$$

6 Comparison to Related Literature

We now examine the related literature to identify our contributions as well as commonalities. These broadly fall into two strands: equilibrium search theory and bargaining on a deadline.

Stigler (1961) initiated the formal modeling of search behavior, but only modeled buyer behavior, taking as given the distribution of prices offered by sellers. The goal of equilibrium search theory has been to complete the model, so that sellers also behave optimally, given the search strategy of buyers. Diamond (1971) highlights the difficulty in sustaining more than one price in equilibrium when both buyers and sellers behave optimally. Successful price dispersion models typically rely on heterogeneous search costs or valuations.⁶

Salop and Stiglitz (1976), Wilde and Schwartz (1979), and Rob (1985) show that a buyer with a higher search cost will settle for a higher price (when encountered), because the expected benefit of another search does not justify the cost. Thus, if buyers differ in their search costs, sellers can offer distinct prices in equilibrium, so long as the lower probability of sale offsets the higher revenue per sale. Butters (1977) generates a similar effect, where some consumers randomly receive more advertisements (and thus have lower search costs) than others.

Alternatively, multiple prices can be sustained when buyers differ in the value they place on the good, and when search involves delaying the acquisition. Diamond (1987) shows that while additional search can eventually produce a lower price, a buyer with a high valuation will settle for a higher price (when encountered) to begin enjoying the good sooner.

In our model, all buyers are ex-ante identical in their valuation of the good and the cost of search (*e.g.* delay). Of course, heterogeneity is necessary to justify sellers offering different prices, but it arises ex-post as some buyers experience longer search spells than others. Luck plays a role in creating these different experiences, as a buyer may not encounter an opportunity to buy; but the buyer is responsible as well, as he chooses the reservation price strategy that may lead him to pass on high prices offered early in his search. These differences in buyer types are endogenously created by adding a very natural feature of deadlines to the search environment.

We also note that a large parallel literature has applied equilibrium search environments to labor markets⁷ with the intent to study wage formation and the effects

⁶The exception comes from Burdett and Judd (1983) in an environment of simultaneous search (where the buyer chooses a number of quotes to seek from random firms, and then selects the best price among those quotes). Even when buyers are identical, simultaneous search can produce a mixed-strategy equilibrium in which buyers randomize between searching once or twice, and sellers randomize over a continuous distribution of prices.

⁷This literature is well surveyed in Rogerson, *et al* (2005).

of unemployment insurance. The mechanisms mentioned above also can sustain wage dispersion; Burdett and Mortensen (1998) add on-the-job search as an additional mechanism. That is, workers continue to receive job offers while employed, but can only be enticed to switch jobs if the new wage exceeds their current wage. This is noteworthy because it can sustain wage dispersion even with ex-ante identical workers.

The job search process is also influenced by deadlines, which are naturally imposed by most unemployment insurance systems by cutting off benefits after a fixed period of time. In Akin and Platt (2011), we adapt our current model to fit this institutional detail, demonstrating that this feature alone can create wage dispersion among ex-ante identical workers. The model and extensions that we study here introduce greater generality, allowing for broader application in any market and providing several variations on how the deadline is modeled (*i.e.* whether the new good is used on purchase or only after the deadline, and whether prior contracts impose early termination fees or post-expiration rate increases).⁸

Next, we turn to the literature on bilateral bargaining that must be accomplished before a deadline. The closest analog to our environment is found in Fuchs and Skrzypacz (2011) (FS), which in turn builds on Sobel and Takahashi (1983) (ST). In both models, the seller does not know the buyer's valuation and makes repeated offers (with an exogenous delay in between each) until the buyer accepts. After an exogenous deadline has passed, both parties receive zero payoff in ST, or an exogenous fraction of the buyer's valuation in FS. The latter also focuses on the limit equilibrium, where the exogenous delay approaches zero. In addition to the possible loss of surplus after the deadline, discounting also motivates both parties to complete the transaction earlier if possible.

The equilibrium strategy has the seller make a decreasing sequence of price offers, and only a buyer with the highest valuations would be willing to accept initial offers. If these early offers are rejected, the buyer is revealed to have a lower valuation; furthermore, the deadline is closer. Thus the seller offers successively lower prices. In spite of the lost surplus, a strictly positive fraction of trades (when the buyer has lower valuations) occur after the deadline has passed.

⁸The comparative statics in our wage formation model also differ in several cases. For instance, a longer deadline increases wages by enabling workers to search over a longer spell, while in our current model, a longer deadline has either no effect on prices (in the late equilibrium) or increases them. That is, a longer grace period can actually harm buyers, while it strictly helped workers.

There are three important differences from our environment: first, the buyer's valuation is not known but the deadline is. We reverse these, so that the seller cannot observe the time until deadline, but knows the buyer's valuation of the good (as well as the post-deadline penalty). Second, one seller and one buyer are engaged in repeated interaction, rather than a one-shot encounter. In our environment, a seller cannot learn anything about the buyer's state until it is too late; we assume there is no opportunity for that seller to make a second offer to that buyer. Third, our seller does not directly face a deadline (she can produce at the same cost at any point), and only cares about the (unknown) deadline of her current customer because it affects his willingness to pay. If the offer is not accepted, however, the next customer's deadline could be very different.

In both their bargaining and our search environments, some buyers will refuse offers in equilibrium. Indeed, with positive probability, some buyers will not make a purchase before the deadline. The conclusions also differ in key ways. In ST or FS, buyers who accept an early offer pay the highest prices; in ours, they pay the lowest prices. The FS limit equilibrium has a continuum of transaction prices, including an atom on the lowest price; our search environment produces an atom at both the lowest and highest price, with a continuum in between.

An important parallel between our comparative statics and those derived in a special case of FS is that as the post-deadline utility increases, fewer transactions occur before the deadline. However, in our model, this causes more transactions take place after the deadline, where in FS, more transactions never occur.

Ma and Manove (1993) is relevant as well. There, the agents make alternating offers on how to split a commonly known surplus, which evaporates after a known deadline. When an agent makes an offer, he can strategically delay how quickly the offer is received (giving the other agent less time to reject and make a counter-offer); at the same time, nature adds a random delay on top of this, creating the risk that the deadline will be crossed. In equilibrium, early offers are rejected, agreements are reached late in the game, and the deadline is missed with positive probability.

Our model serves as a bridge between the literature on bargaining on a deadline and equilibrium search, but admittedly other assumptions could be used. The only existing alternative is found in Albrecht, *et al* (2007) (AASV). As in our model, agents enter the market in a relaxed state (with a high instantaneous utility); but eventually they switch to a desperate state (with a low instantaneous utility). The

key difference is that this transition occurs randomly, at a given Poisson rate, rather than deterministically after a fixed period of time. Thus, time is not a state variable. In addition, they assume that buyers and sellers can observe each others' states on meeting, and then determine prices via Nash bargaining. In our incomplete information environment, sellers must make a probabilistic judgment as to the likelihood of its product being purchased when it posts a particular price.⁹

As a consequence, our model generates different comparative statics. For instance, a longer deadline has no effect on prices in our model, as the search process is merely delayed by an equal amount of time. In AASV, a lower Poisson rate (hence longer expected duration until desperation sets in) will decrease prices because successful matches are more likely to occur while in the relaxed state; also the range of prices shrinks. Another stark difference is when the value x of the good to buyers increases, we see the average price fall and the range of prices increase, while AASV find the opposite. These and other differences are mostly driven by the deterministic versus random transition. To verify this, we set up our model in an environment where buyers and sellers randomly transition from relaxed to desperate, but sellers have no information on the state of buyers; thus relaxed sellers charge a price that is only acceptable to desperate buyers, and desperate sellers charge a price that any buyer will accept (opportunistic matching). This yields a simple solution, whose comparative statics closely follow AASV.

⁹Although there are many examples where the impending deadline is obvious, there are equally plausible scenarios in which it is not apparent and cannot be credibly communicated. For instance, most home buyers will not want to reveal themselves to be in a hurry (such as a job start date or upcoming birth) until the home is under contract, to avoid price discrimination by the seller. When a consumer searches for a new cell phone or cable TV provider, the (potential) new providers would be unsure about how close the consumer is to completing his current contract. Typically, they resort to posted prices; occasional discount offers can be seen as a mixed strategy. Similarly, if a borrower is seeking to refinance his home or transfer credit card balances before an interest rate increase, the new banks will not typically be privy to the timeline of this rate reset.

At the same time, we cannot dismiss bargaining as an important aspect in many markets. For instance, a seller might list her house at a high asking price, anticipating that bargaining will chisel to a lower eventual sales price. Even so, Horowitz (1992) estimates a model of price formation to the Baltimore housing market and finds that there is a one-to-one relationship between asking- and sales-prices. Thus, price posting might still be a good approximation, since sellers can anticipate that listing a higher asking price will result in a higher sales price (but a longer expected time before sale).

7 Conclusion

We analyze price formation in a market where buyers face a deadline to complete their transaction. Sellers are uninformed about how much time a potential buyer has remaining until his deadline; therefore, a higher posted price increases the profit if sold, but reduces the probability of sale. In this incomplete information environment, we solve for the endogenous price distribution that prevails in the steady state. We do this by translating the equilibrium conditions of the dynamic search problem into a second-order differential equation, which has a unique solution.

The equilibrium exhibits many interesting properties. First, buyers initially forgo any purchases, preferring to delay the purchase till they get closer to the deadline. As the deadline approaches, their reservation price rises, making it possible for transactions to occur. Second, the distribution of seller's asking prices includes two atoms. One is at the highest price, which is only accepted by the unlucky buyers who reached the deadline. The other is at the lowest price, targeting those who have just entered the market.

The comparative statics reveal some surprising results. For instance, as the grace-period utility increases, buyers delay their purchases even longer. Sellers exploit this delay by raising prices, which discourages early acceptance. On the other hand, an increase in the value of the good actually causes prices to fall. Here, buyers are motivated to obtain the good earlier, which reduces the market power of sellers. Finally, increasing the length of the grace period has no effect on the equilibrium outcome; buyers still delay their purchases until the same point, and prices are unaltered.

This search framework has many potential applications, such as the study of marriage markets. Female fertility declines with age, imposing a natural deadline in a woman's search process; male fertility remains roughly constant. Thus, a woman's search process could be characterized by a reservation quality (of spouse), which would decline as the deadline approaches.

In addition, the model presented here can easily be adapted to understand how deadlines on the *seller* affect the pricing of a commodity over time. For instance, Copeland, *et al* (2011) present evidence that in the automobile market, manufacturers reduce the price of an automobile model over time, as they feel the pressure to eliminate inventory before next year's model is introduced to the market. The model release cycle induces a deadline, after which the prior model will drop in value.

A Proofs

A.1 Proposition 1

Proof. The translation process demonstrated in the text ensures that the $R(s)$ function satisfies conditions 1 and 3 in the equilibrium definition, and we can see in Eq. 20 that all prices in the support of $F(p)$ are equally profitable. We only need to verify that any price outside the support generates weakly lower profits. Of course, any price above $R(0)$ will be rejected by all buyers and hence generates zero profit. So we now inspect the impact of offering a price below $R(S^*)$, taking as given $H(s)$ and $R(s)$. We proceed assuming that this is a dispersed equilibrium; the same logic applies in a degenerate equilibrium, after appropriately substituting the equations used.

First, we must compute the equilibrium reservation prices of buyers in states $s \in (S^*, T]$. While none of these prices occur in equilibrium, one can compute the hypothetical price at which the buyer is indifferent between purchasing and continuing search, using $R(s) = \frac{x}{\rho} - V(s)$. In these states, the Bellman equation would be $\rho V(s) = b - V'(s)$, with the boundary condition $V(S^*) = \frac{x}{\rho} - R(S^*)$. We then solve for $V(s)$ in this first-order differential equation, and substitute to find $R(s)$, obtaining:

$$R(s) = \frac{x - b + (b - x + \rho c)e^{-\rho(s-S^*)}}{\rho} + \frac{b - d}{\lambda} e^{-\frac{2\lambda}{\rho}\left(1 - e^{-\frac{\rho S^*}{2}}\right) - \rho(s-S^*)}.$$

The fraction of buyers willing to accept such an offer is $H(s) = \frac{\lambda(s-S^*) + e^{\frac{\rho S^*}{2}}}{\lambda(T-S^*) + e^{\frac{\rho S^*}{2}}}$. Thus, the expected profit from offering a price targeted for s is $(R(s) - c)\frac{H(s)}{H(T)}$, which becomes:

$$\Pi(s) \equiv \left(\frac{(x - b - \rho c)(1 - e^{-\rho(s-S^*)})}{\rho} + \frac{b - d}{\lambda} e^{-\frac{2\lambda}{\rho}\left(1 - e^{-\frac{\rho S^*}{2}}\right) - \rho(s-S^*)} \right) \frac{\lambda(s - S^*) + e^{\frac{\rho S^*}{2}}}{\lambda(T - S^*) + e^{\frac{\rho S^*}{2}}}.$$

At $s = S^*$, $\Pi(S^*)$ provides the same expected profit that is experienced by offering a price anywhere in the support of F . This can be simplified by substituting for

$e^{-\frac{2\lambda}{\rho}\left(1-e^{-\frac{\rho S^*}{2}}\right)}$ using the equation $\phi(S^*) = 0$, to get:

$$\Pi(s) \equiv \left(\lambda (1 - e^{-\rho(s-S^*)}) - \rho e^{\frac{\rho S^*}{2}} \right) \frac{(x - b - \rho c) \lambda (s - S^*) + e^{\frac{\rho S^*}{2}}}{\rho \left(\lambda - \rho e^{\frac{\rho S^*}{2}} \right) \lambda (T - S^*) + e^{\frac{\rho S^*}{2}}}.$$

Its first derivative is:

$$\Pi'(s) = \frac{\lambda(x - b - \rho c) \left(\left(\lambda - \rho e^{\frac{\rho S^*}{2}} \right) (1 - e^{-\rho(s-S^*)}) + \lambda \rho (s - S^*) e^{-\rho(s-S^*)} \right)}{\rho \left(\lambda - \rho e^{\frac{\rho S^*}{2}} \right) \left(\lambda (T - S^*) + e^{\frac{\rho S^*}{2}} \right)}.$$

When evaluated at $s = S^*$, this derivative is equal to zero (due to the last parenthetical term in the numerator). For any $s > S^*$, $\Pi'(s) < 0$. To see this, first note that $\phi(S^*) = 0$ implies that $x - b - \rho c < 0$. By assumption, $\lambda - \rho e^{\frac{\rho S^*}{2}} > 0$; thus all other terms are positive.

Thus, if a seller were to offer any price just below $R(S^*)$, his profits would be no greater than the equilibrium π . Moreover, as he offers even lower prices (*i.e.* targets a larger s), his profit is strictly decreasing. Thus deviation from the equilibrium pricing strategy is not profitable.

Thus, the proposed equilibrium satisfies all the necessary conditions for equilibrium.

□

A.2 Early Equilibria

When benefits are not sufficiently generous (*i.e.* when $\phi(T) > 0$), the equilibrium presented in Section 3 does not exist. However, there is another equilibrium solution, which we refer to as *early equilibria* because consumers begin accepting at least some prices that they encounter immediately on entering the market (*i.e.* $R(T)$ is in the support F). This is in contrast with the *late equilibria* depicted in Section 3, where consumers wait until later in their spell before any of the offers are acceptable.

First, it is possible that $R(T)$ is the only price offered in an *early degenerate equilibrium*. In this case, it is quite simple to solve the Bellman equations and the steady state equations, as they each constitute a first-order differential equation,

whose solution is:

$$V(s) = \frac{b}{\rho} - \frac{b-d}{\rho(\rho+\lambda)} (\lambda e^{-(\rho+\lambda)T} + \rho e^{-(\rho+\lambda)s}) \quad (30)$$

$$H(s) = \frac{\delta}{\lambda} e^{-\lambda(T-s)}, \quad (31)$$

with the only price offered in equilibrium being $R(T) = \frac{1}{\rho} (x - b + (b - d)e^{-(\rho+\lambda)T})$. The only issue is to verify that it is not profitable for sellers to deviate, offering prices $R(s)$ for $s < T$. Using the same approach as in Appendix A.1, one can show that deviation is unprofitable if and only if the following condition holds:

$$(b-d)e^{-T(\rho+\lambda)} \leq \frac{\lambda}{\lambda-\rho} (b-x+c\rho). \quad (32)$$

It is also possible to have early dispersed equilibria, where the support of F will span from $R(T)$ to $R(Z^*)$ for some $Z^* \in [0, T]$. The distribution F will include atoms at $R(T)$ and $R(Z^*)$. The method to reach this solution closely follows that used in Section 3.

For instance, in translating the Bellman equation for $s \in [Z, T]$, we still obtain:

$$(\rho + \lambda F(R(s))) R'(s) = -R''(s). \quad (33)$$

Moreover, the requirement for continuity at $s = T$ results in:

$$x - b - \rho R(T) = R'(T), \quad (34)$$

mirroring Eq. 10. By the same token, the translated steady state conditions for the interior of the support of $F(p)$ and for $s = T$ carry over similarly:

$$F(R(s)) = \frac{R''(s) - \frac{2R'(s)^2}{R(s)-c}}{\lambda R'(s)} \quad (35)$$

$$R'(T) = -\frac{\delta(R(T)-c)}{H(T)}. \quad (36)$$

The only difference occurs for the boundary conditions at $s = 0$. For instance, the Bellman equation still requires $V'(0) = b - d$ to maintain continuity. However, for $s \in [0, Z)$, the distribution of acceptable offers remains constant (since no prices

above $R(Z^*)$ are offered):

$$(\rho + \lambda)V(s) = b - V'(s) + \lambda \int_{-\infty}^{R(Z^*)} \left(\frac{x}{\rho} - p \right) dF(p). \quad (37)$$

For s in this interval, the Bellman equation is a first-order differential equation, with boundary condition $V'(0) = b - d$. This has the unique solution:

$$(\rho + \lambda)V(s) = (d - b)e^{-(\rho+\lambda)s} + b + \lambda \int_{-\infty}^{R(Z^*)} \left(\frac{x}{\rho} - p \right) dF(p). \quad (38)$$

By substituting for the integral by using Eq. 37, we get: $V'(s) = (b - d)e^{-(\rho+\lambda)s}$. We then evaluate this at $s = Z^*$ and substitute with $V'(Z^*) = -R'(Z^*)$:

$$R'(Z^*) = (d - b)e^{-(\rho+\lambda)Z^*}. \quad (39)$$

Thus, the continuity condition at $s = 0$ becomes a boundary condition on our differential equation 33 at $s = Z^*$.

A similar translation occurs for the steady state conditions. Since every proposed offer is accepted by those with $s \leq Z^*$ time remaining, Eq. 5 becomes $H''(s) = \lambda H'(s)$. To solve this second-order differential equation, we need two boundary conditions. One comes from the continuity condition at $s = 0$; that is, $H'(0) = \lambda H(0)$. The other comes from the equal profit condition, comparing expected profits from offering price $R(T)$ to that from $R(Z^*)$, which yields: $H(Z^*)(R(Z^*) - c) = H(T)(R(T) - c)$. The solution of this system is:

$$H(s) = \frac{R(T) - c}{R(Z^*) - c} \lambda H(T) e^{\lambda(s - Z^*)}. \quad (40)$$

Taking the derivative w.r.t. s , evaluated at $s = Z^*$, we get $H'(Z^*) = \frac{R(T) - c}{R(Z^*) - c} \lambda H(T)$.

At the same time, the constant profit condition requires: $H(T)(R(T) - c) = H(s)(R(s) - c)$ for all $s \in [Z^*, T]$. Its derivative w.r.t. s , evaluated at $s = Z^*$, is $H'(Z^*) = -\frac{H(T)(R(T) - c)R'(Z^*)}{(R(Z^*) - c)^2}$. These two derivatives (approaching Z^* from the left or the right) must equate so that the buyers' population density remains continuous. This requirement simplifies to:

$$R'(Z^*) = \lambda(c - R(Z^*)). \quad (41)$$

With this, we are prepared to solve the second-order differential equation of $R(s)$ for $s \in [Z^*, T]$. Indeed, this differential equation is unchanged: $R'(s) \left(\rho + \frac{2R'(s)}{c-R(s)} \right) + 2R''(s) = 0$. Using Eq. 39 and 41 as boundary conditions, the unique solution is:

$$R(s) = c + \frac{b-d}{\lambda} e^{-(\rho+\lambda)Z^* - \frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho}{2}(s-Z^*)}\right)} \quad \text{for } s \in [Z^*, T]. \quad (42)$$

Eq. 36 pins down $H(T)$, so that:

$$H(s) = \begin{cases} \frac{\delta}{\lambda} e^{\frac{\rho}{2}(T-Z^*) + \frac{2\lambda}{\rho} \left(e^{-\frac{\rho}{2}(T-Z^*)} - e^{-\frac{\rho}{2}(s-Z^*)} \right)} & \text{if } Z^* \leq s \leq T \\ \frac{\delta}{\lambda} e^{\frac{\rho}{2}(T-Z^*) + \lambda(s-Z^*) + \frac{2\lambda}{\rho} \left(e^{-\frac{\rho}{2}(T-Z^*)} - 1 \right)} & \text{if } 0 \leq s < Z^*. \end{cases} \quad (43)$$

Eq. 34, on the other hand, is used to determine Z^* :

$$\psi(Z^*) \equiv \frac{x-b}{\rho} - c + \frac{b-d}{\rho} \left(e^{-\frac{\rho}{2}(T-Z^*)} - \frac{\rho}{\lambda} \right) e^{-(\rho+\lambda)Z^* - \frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho}{2}(T-Z^*)}\right)}. \quad (44)$$

If there exists a $Z^* \in [0, T)$ such that $\psi(Z^*) = 0$, then this constitutes a dispersed equilibrium. We omit the proof of this, as it precisely follows that in Appendix A.1. Having obtained the preceding solution, it is a simple algebraic exercise to compute the remaining equilibrium objects. The price distribution in the early dispersed equilibrium is:

$$F(p) = \begin{cases} 0 & \text{if } p < R(T) \\ 1 - \frac{\rho}{2\lambda} \left((\rho + \lambda)Z^* - \ln \left[\frac{b-d}{\lambda(p-c)} \right] \right) & \text{if } R(T) < p < R(Z^*) \\ 1 & \text{if } p \geq R(Z^*), \end{cases} \quad (45)$$

or expressed in terms of which buyers are being targeted by a particular seller:

$$F(R(s)) = \begin{cases} e^{\frac{\rho}{2}(Z^*-s)} - \frac{\rho}{2\lambda} & \text{if } Z^* < s \leq T \\ 1 & \text{if } s < Z^*. \end{cases}$$

The sellers earn expected profit:

$$\pi = \frac{\delta(b-d)}{\lambda} e^{\frac{\rho(T-Z^*)}{2} - (\rho+\lambda)Z^* - \frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho}{2}(T-Z^*)}\right)} \quad \text{if } p \in [R(T), R(Z^*)]. \quad (46)$$

The qualitative features of this early equilibrium, such as the shape of the reser-

vation price function and price distribution, are nearly the same as those of the late equilibrium.

A.3 Proposition 2

Proof. First, consider possible late equilibria. The first derivative of ϕ is:

$$\phi'(S) = \frac{b-d}{2\rho} \left(\rho e^{\frac{\rho S}{2}} - 2\lambda \right) e^{-\frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho S}{2}} \right) - \rho S}. \quad (47)$$

Since we have assumed that $b > d$ and $e^{-\frac{\rho T}{2}} > \frac{\rho}{2\lambda}$, then $\rho e^{\frac{\rho S}{2}} < 2\lambda$ and hence $\phi'(S) < 0$ for all $S \leq T$.

Thus, if $\phi(0) \leq 0$, then $\phi'(S) < 0$ implies that $\phi(S) < 0$ for all $S \in (0, T]$. If so, the late degenerate equilibrium exists.

On the other hand, if $\phi(0) > 0$ and $\phi(T) \leq 0$, then the continuity of $\phi(S)$ implies that there exists an $S^* \in [0, T]$ such that $\phi(S^*) = 0$. Moreover, since $\phi'(S) < 0$, then S^* is unique.

If instead $\phi(T) > 0$, then $\phi'(S) < 0$ implies that $\phi(S) > 0$ for all $S \in [0, T]$. Thus neither a late degenerate nor a late dispersed equilibrium can occur.

Similarly, consider possible early equilibria. The first derivative of ψ is:

$$\psi'(Z) = \frac{b-d}{2\lambda\rho} \kappa(Z) e^{-(\rho+\lambda)Z - \frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho}{2}(T-Z)} \right)}, \quad (48)$$

where

$$\kappa(Z) = 2\rho^2 - \lambda\rho e^{-\frac{\rho}{2}(T-Z)} + 2\lambda \left(\rho - \lambda e^{-\frac{\rho}{2}(T-Z)} \right) \left(1 - e^{-\frac{\rho}{2}(T-Z)} \right). \quad (49)$$

Recall that we have assumed throughout that $e^{-\frac{\rho T}{2}} > \frac{\rho}{\lambda}$. Thus, $\rho < \lambda e^{-\frac{\rho T}{2}} \leq \lambda e^{-\frac{\rho}{2}(T-Z)}$; and since $1 > e^{-\frac{\rho}{2}(T-Z)}$, the last term of $\kappa(Z)$ is always negative.

The stronger assumption stated in the proposition ensures that the first two terms also are negative, because:

$$\lambda\rho e^{-\frac{\rho}{2}(T-Z)} \geq \lambda\rho e^{-\frac{\rho T}{2}} > 2\rho^2.$$

Thus $\kappa(Z) < 0$, and since the other terms in $\psi'(Z)$ are positive, then $\psi'(Z) < 0$.

Thus, if $\psi(T) \geq 0$, then $\phi(Z) > 0$ for all $Z \in [0, T)$. If so, the early degenerate

equilibrium exists.

On the other hand, if $\psi(T) < 0$ and $\psi(0) \geq 0$, then the continuity of $\psi(Z)$ implies that there exists an $Z^* \in [0, T]$ such that $\psi(Z^*) = 0$. Moreover, since $\psi'(Z) < 0$, then Z^* is unique.

If instead $\psi(0) < 0$, then $\psi(Z) < 0$ for all $Z \in [0, T]$, and neither an early degenerate nor an early dispersed equilibrium can occur.

Finally, note that $\phi(T) = \psi(0)$. Thus, if $\psi(0) \leq 0$ (so neither early equilibrium exists), then $\phi(T) \leq 0$ and thus one of the late equilibria exists. Similarly, if $\phi(T) > 0$ (so neither late equilibrium exists), then $\psi(0) > 0$ and one of the early equilibria exists. \square

A.4 Lemma 1

Proof. In the proof of Proposition 2 (Section A.3), we established that if a dispersed equilibrium exists, then $\frac{\partial \phi}{\partial S} < 0$ or $\frac{\partial \psi}{\partial Z} < 0$. In addition, the partial derivatives with respect to d , x , λ , and T are:

$$\begin{aligned} \frac{\partial \phi}{\partial T} &= 0 \\ \frac{\partial \psi}{\partial T} &= (b-d) \left(\frac{1}{2} - \frac{\lambda}{\rho} e^{-\frac{\rho(T-Z^*)}{2}} \right) e^{-(\rho+\lambda)Z^* - \frac{\rho(T-Z^*)}{2} - \frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho(T-Z^*)}{2}} \right)} < 0 \\ \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial x} = \frac{1}{\rho} > 0 \\ \frac{\partial \phi}{\partial c} &= \frac{\partial \psi}{\partial c} = -1 < 0 \\ \frac{\partial \phi}{\partial d} &= \left(\frac{1}{\lambda} - \frac{e^{-\frac{\rho S^*}{2}}}{\rho} \right) e^{-\frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho S^*}{2}} \right)} < 0 \\ \frac{\partial \psi}{\partial d} &= \left(\frac{1}{\lambda} - \frac{e^{-\frac{\rho(T-Z^*)}{2}}}{\rho} \right) e^{-(\rho+\lambda)Z^* - \frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho(T-Z^*)}{2}} \right)} < 0 \\ \frac{\partial \phi}{\partial b} &= -\frac{1}{\rho} \left(1 - \left(e^{-\frac{\rho S^*}{2}} - \frac{\rho}{\lambda} \right) e^{-\frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho S^*}{2}} \right)} \right) < 0 \\ \frac{\partial \psi}{\partial b} &= -\frac{1}{\rho} \left(1 - \left(e^{-\frac{\rho(T-Z^*)}{2}} - \frac{\rho}{\lambda} \right) e^{-(\rho+\lambda)Z^* - \frac{2\lambda}{\rho} \left(1 - e^{-\frac{\rho(T-Z^*)}{2}} \right)} \right) < 0. \end{aligned}$$

To obtain the sign on $\frac{\partial \phi}{\partial d}$, recall that by assumption, $\frac{\rho}{\lambda} < e^{-\frac{S^*}{2}}$. For $\frac{\partial \phi}{\partial b}$, note that

$1 > e^{-\frac{\rho S^*}{2}} - \frac{\rho}{\lambda} > 0$, and the last exponential term will also be less than 1.

Thus, by implicit differentiation, $\frac{\partial S^*}{\partial b} = -\frac{\partial \phi / \partial b}{\partial \phi / \partial S} < 0$, and similarly, $\frac{\partial S^*}{\partial x} > 0$, $\frac{\partial S^*}{\partial c} < 0$, $\frac{\partial S^*}{\partial d} < 0$, and $\frac{\partial S^*}{\partial T} = 0$. The same occurs for derivatives of ψ : $\frac{\partial Z^*}{\partial b} < 0$, $\frac{\partial Z^*}{\partial x} > 0$, $\frac{\partial Z^*}{\partial c} < 0$, $\frac{\partial Z^*}{\partial d} < 0$, and $\frac{\partial Z^*}{\partial T} < 0$. \square

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