Equal sacrifice taxation*

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Abstract

We characterize the family of equal sacrifice rules for the problem of fair taxation: every individual with positive post-tax income sacrifices the same amount of utility relative to his/her respective pre-tax income. Because we do not impose Symmetry or Strict Resource Monotonicity in our set of axioms, our family of rules allows for asymmetric and "constrained" versions of equal sacrifice. In addition, we show that when Linked Claims-Resource Monotonicity is added to the set of axioms, then this is equivalent to adding the requirement that every individual's utility function is concave.

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Equality of taxation, therefore, as a maxim of politics, means equality of sacrifice. It means apportioning the contribution of each person towards the expenses of government, so that he shall feel neither more nor less inconvenience from his share of the payment than every other person experiences from his. This standard, like other standards of perfection, cannot be completely realized; but the first object in every practical discussion should be to know what perfection is.

–Mill, *Principles of Political Economy*

1 Introduction

Consider the problem of fair taxation: Given a fixed amount of tax revenue that needs to be raised, and given the amount of income that each citizen has, how much should each citizen be taxed? This has long been a problem of interest to philosophers, economists, and politicians. Indeed, discussions of fair taxation in the public sphere often follow any proposal to modify the tax system.

One proposed method of fair taxation is to impose an equal amount of subjective sacrifice on each individual. The idea of equal sacrifice as a notion of fairness can be traced back to John Stuart Mill. The first axiomatic study of the equal sacrifice principle applied to fair taxation is Young (1988). In that paper, Young considers the family of symmetric and unconstrained equal sacrifice taxation methods (called rules). A member of this family is defined by a utility function U which is continuous, strictly increasing, and unbounded from below. The rule then chooses taxes owed by each individual so that each individual's utility loss is the same. That is, for individuals i and j with pre-tax incomes c_i and c_j and post-tax incomes x_i and x_j, we have

$$U(c_i) - U(x_i) = U(c_j) - U(x_j).$$

Such a rule is symmetric in the sense that the same U is applied to all individuals. The rule is unconstrained in the sense that it is always able to equalize sacrifice across all individuals;[1] there is never an instance in which the rule must impose less sacrifice on an individual because it is impossible for the individual to pay more in taxes then her income.

In this paper, we consider a more general family of equal sacrifice rules, one that allows for both asymmetric and constrained rules. A member of this family is defined by a collection of utility functions $\{U_i\}$, one for each individual, where each U_i is continuous and strictly increasing (but not necessarily unbounded from below). The rule then chooses taxes owed by each individual so that each individual receiving strictly positive post-tax income will have the same utility loss. That is, for individuals i and j with pre-tax incomes c_i and c_j and post-tax incomes $x_i > 0$ and $x_j > 0$, we have

$$U_i(c_i) - U_i(x_i) = U_j(c_j) - U_j(x_j).$$

[1]This is due to the fact that U is unbounded from below. Thus for any utility loss λ, one can always find a post-tax income level x_i such that $U(c_i) - U(x_i) = \lambda$.

This allows for asymmetric rules since U_i and U_j are potentially different utility functions. In addition, the rule may be constrained in the sense that one individual may experience less sacrifice than the others because the rule cannot assign a tax more than an individual's income. In this case, the individual's assigned tax would be equal to her income (leaving zero post-tax income), and thus we may have

$$U_i(c_i) - U_i(0) < U_j(c_j) - U_j(x_j).$$

Our main theorem, Theorem 1, axiomatically characterizes this family of rules. Our two most important axioms are prominent in the literature. The first, Consistency, says that how a rule assigns taxes does not change when the group to be taxed shrinks coupled with an appropriate shrinking of the tax burden. The second, Composition Down, says that if the total tax burden increases, then it is sufficient to use current income (i.e. the post-tax income under the previous, smaller tax burden) to determine the new assignment of taxes.

In addition, we impose a novel axiom which is weaker than Strict Claims Monotonicity, a well-known axiom in the literature. Strict Claims Monotonicity states that if one individual's income increases, then her post-tax income should increase. We impose this requirement as well, but only in instances in which everyone currently has positive post-tax income. We call our axiom Lower Constrained Strict Claims Monotonicity.

One common axiom obviously missing from this result is Symmetry, which states that two individuals with equal income will be taxed equally. Theorem 3 shows that when we add Symmetry to our set of axioms, the result is a generalization of Young's family that allows for constrained rules. Thus one contribution of this paper is simply a better understanding the logical implications of Symmetry. That is, because Symmetry invariably plays a central role in the proof of any theorem that employs it, an important question is what happens without it. Theorem 1 and Theorem 3 together demonstrate that relaxing Symmetry (in the presence of our other axioms) does nothing more than allow for different utility functions for the agents.[2]

Given the prominence of concave utility functions in economic theory, a natural question is what implications concavity would have on our division rule. Theorem 4 shows that adding an axiom called Linked Claims-Resource Monotonicity to the set of axioms from Theorem 1 is equivalent to adding the requirement that the utility functions $\{U_i\}$ all be concave. In the context of taxation, Linked Claims-Resource Monotonicity is simply the requirement that when one individual's pre-tax income increases, then her tax burden must weakly increase.

Allowing for asymmetric equal sacrifice rules is a natural extension of Young's family of rules. However, allowing for asymmetry may also be desirable for normative reasons. That is, for reasons of fairness, the taxing authority may want to treat two individuals differently simply because they have different needs and situations. Indeed, in the United States, one's tax burden is determined by more than just pre-tax income, such as the number of dependents the individual has. Given this observation,

[2] See Stovall (2014a) for an example in which relaxing Symmetry does not lead to a straightforward result.

one approach would be to extend the model to include all relevant information the taxing authority uses to determine the assignment of taxes. However, to keep the model broadly applicable to other contexts, as well as to make it easily comparable to the existing literature, we keep the standard framework wherein only identities and pre-tax income are used to determine the assignment of taxes.

Allowing for constrained versions of equal sacrifice rules is desirable given that the most natural form of equal sacrifice is to simply impose the same tax on each individual. However, to be valid, the rule must be constrained since someone cannot be taxed more than their income. In the literature, this is referred to as the constrained equal loss rule, though in the context of taxation it is commonly called the head tax. Young's family of rules excludes this central rule. However, the head tax is a member of our family of rules.

Formally, the problem of fair taxation is identical to the problem of fair allocation under conflicting claims: Given a fixed amount of a resource that must be divided among a group, each individual of the group having some (objective) claim on the resource, and given that the amount to be divided is not sufficient to satisfy all claims, how should the resource be divided? Other examples of conflicting claims problems are bankruptcy and cost sharing. Modern study of claims problems began with O'Neill (1982). See Thomson (2003, 2015) for surveys of this literature. We consider the present work part of this literature, and thus borrow much of its terminology (e.g. in naming axioms, we use 'resource', 'claims', and 'award' instead of 'tax burden', 'pre-tax income', and 'post-tax income', respectively). We discuss this further in section 2.

Besides Young's paper, there are two other papers in the literature closely related to the present work. Chambers and Moreno-Ternero (2017) consider a generalized family of symmetric equal sacrifice rules that allows for constrained versions of Young's family. Naumova (2002) considers asymmetric equal sacrifice rules, but only ones that are unconstrained. More broadly, the current work (like Naumova's) adds to the growing literature studying asymmetric rules for the claims problem: Chambers (2006), Hokari and Thomson (2003), Kıbrıs (2012, 2013), Moulin (2000), and Stovall (2014a,b) all consider rules that are (possibly) asymmetric. Of these papers, only Stovall (2014a) is easily relatable to the family we characterize. We discuss these related papers in more detail in section 4. For readers wishing to preview the relation between these other papers and the current work, Table 1 and Figure 1 provide summaries.

2 The Model

We use the following notation. Let $\mathcal{N}$ denote the set of finite subsets of the natural numbers, $\mathbb{N}$. Let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the non-negative real numbers and the positive real numbers respectively. Let $\mathbf{0}$ denote a vector of zeros. For $x, y \in \mathbb{R}^N$, we use the vector inequalities $x \geqq y$ if $x_i \geq y_i$ for all $i \in N$, $x \geq y$ if $x \geqq y$ and $x \neq y$, and $x > y$ if $x_i > y_i$ for every $i \in N$. For $x \in \mathbb{R}^N$ and $N' \subset N$, let $x_{N'}$ denote the projection of x onto the subspace $\mathbb{R}^{N'}$. For $i \in N$, let x_{-i} denote $x_{N\setminus\{i\}}$.

A *problem* is a tuple (N, c, E) where $N \in \mathcal{N}$, $c \in \mathbb{R}^N_{++}$, and $E \in [0, \sum_i c_i]$. An *award* for the problem (N, c, E) is an N-vector x satisfying $0 \leqq x \leqq c$ and $\sum_i x_i = E$. A *rule* is a function S that maps problems to awards.

In the context of taxation, we think of c_i as being agent i's pre-tax income and E as representing the total amount of post-tax income. Thus $\sum_i c_i - E$ is the total amount of tax to be collected. The requirement $E \leq \sum_i c_i$ says that the amount of tax to be raised is positive, while the requirement $E \geq 0$ says that it does not exceed national income. An award x_i for agent i is the amount of post-tax income that i gets. Thus the requirement $x_i \geq 0$ says that an agent cannot be taxed more than her income, while the requirement $x_i \leq c_i$ says that an agent's income cannot be subsidized by tax revenue. Finally the requirement that $\sum_i x_i = E$ combines the feasibility requirement ($\sum_i x_i \leq E$) and the efficiency requirement ($\sum_i x_i \geq E$).

As mentioned in the introduction, a taxation problem is formally equivalent to the problem of fair allocation under conflicting claims, the most prominent example of such a problem being bankruptcy. In this context, c_i is agent i's claim on the resource, while E is the total amount of the resource to be divided. The requirement that $E \leq \sum_i c_i$ says that there is not enough of the resource to satisfy everyone's claim on it. We usually think of E as representing a resource that is desirable for all agents, though this is not necessary. Indeed, an alternate way of thinking about a problem is not how to divide the resource, but rather how to divide the loss among the agents. That is, $\sum_i c_i - E$ represents the shortage, or loss, that must be divided. This dual way of thinking about a problem brings us to the following definitions.

Definition. The *dual of a problem* (N, c, E) is the problem $(N, c, \sum_i c_i - E)$. The *dual of a rule* S is the rule S^d satisfying $S^d(N, c, E) = c - S(N, c, \sum_i c_i - E)$ for every problem (N, c, E). The *dual of an axiom* A is the axiom A^d such that S satisfies A if and only if S^d satisfies A^d. An axiom A is *self-dual* if $A^d = A$.

Since our ultimate goal is to study fair taxation, it may seem like a roundabout approach to study rules that allocate post-tax income rather than rules that allocate taxes directly. However, to make our results more readily comparable to the literature on conflicting claims, we adopt the perspective that the resource to be divided, E, is desirable for the agents. Thus E represents the total amount of post-tax income. Given the definitions above, it is a straightforward step to go from studying income allocation rules to studying tax allocation rules. That is, if S is a post-tax income allocation rule that satisfies axiom A, then S^d is a tax allocation rule that satisfies axiom A^d.

We note that Young (1988) takes the opposite approach; i.e. the resource to be divided is the total tax revenue, and is thus undesirable. Therefore any comparison between Young's results and ours should take this duality in to account.

2.1 Equal Sacrifice Rules

Let $\mathcal{U}$ denote the family of functions $U : \mathbb{N} \times \mathbb{R}_{++} \to \mathbb{R}$ such that, for any $i \in \mathbb{N}$, $U(i, \cdot)$ is continuous and strictly increasing. Note that we may or may not have $\lim_{x \to 0} U(i, x) = -\infty$. From now on we write $U(i, \cdot)$ as U_i.

For any $U \in \mathcal{U}$, we define the *equal sacrifice rule relative to* U to be the rule that allocates by equalizing the utility loss of every agent (relative to their pre-tax income) with the proviso that no agent is awarded a negative amount. Hence for any problem (N, c, E), if $i, j \in N$ both get positive awards (say x_i and x_j), then we must have $U_i(c_i) - U_i(x_i) = U_j(c_j) - U_j(x_j)$.

To define these rules formally, we introduce some notation. For every i, set $\underline{u}_i \equiv \lim_{x\to 0} U_i(x)$ and $\bar{u}_i \equiv \lim_{x\to\infty} U_i(x)$. Since U_i is continuous and strictly increasing, it is invertible over $(\underline{u}_i, \bar{u}_i)$. Let $U_i^{-1} : (\underline{u}_i, \bar{u}_i) \to \mathbb{R}_{++}$ denote the inverse function of U_i. Let $\overline{U_i^{-1}}$ denote the left-hand extension of U_i^{-1}, i.e.

$$\overline{U_i^{-1}}(u) \equiv \begin{cases} 0 & \text{if } u \leq \underline{u}_i, \\ U_i^{-1}(u) & \text{if } \underline{u}_i < u < \bar{u}_i. \end{cases}$$

Note that $\overline{U_i^{-1}}$ is continuous, weakly increasing on $(-\infty, \bar{u}_i)$, and strictly increasing on $[\underline{u}_i, \bar{u}_i)$.

For $U \in \mathcal{U}$, we define the rule ES^U as follows. For any problem (N, c, E),

$$ES^U(N, c, E) \equiv \left(\overline{U_i^{-1}}\left(U_i(c_i) - \lambda\right)\right)_{i\in N},$$

where $\lambda \geq 0$ is chosen so that $\sum_{i\in N} \overline{U_i^{-1}}\left(U_i(c_i) - \lambda\right) = E$.[3] We say a rule S is an *equal sacrifice rule* if there exists $U \in \mathcal{U}$ such that $S = ES^U$. We say that $U \in \mathcal{U}$ is an *equal sacrifice representation of* ES^U. We use $\mathcal{ES}$ to denote the family of equal sacrifice rules, i.e.

$$\mathcal{ES} \equiv \left\{ES^U : U \in \mathcal{U}\right\}.$$

The family of equal sacrifice rules contains some prominent rules. The *proportional rule*, P, allocates post-tax income proportionally to pre-tax income; i.e.

$$P(N, c, E) = \frac{E}{\sum_i c_i} c.$$

This is commonly referred to as the *flat tax*, where $1 - \frac{E}{\sum_i c_i}$ is the tax rate. The proportional rule is an equal sacrifice rule where $U_i(x) = U_j(x) = \ln x$ for every $i, j \in \mathbb{N}$.

The *constrained equal loss rule*, CEL, imposes the same loss (i.e. tax) on every individual as long as that tax is not more than their respective income; i.e. for $i \in N$,

$$CEL_i(N, c, E) = \max\{0, c_i - \lambda\},$$

[3] Note that ES^U is well-defined and a rule: For any $i \in \mathbb{N}$, $c_i > 0$, and $\lambda \geq 0$, we must have $0 \leq \overline{U_i^{-1}}\left(U_i(c_i) - \lambda\right) \leq c_i$. Also, note that for any $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, $F(\lambda) \equiv \sum_{i\in N} \overline{U_i^{-1}}\left(U_i(c_i) - \lambda\right)$ is continuous and strictly decreasing, $F(0) = \sum_{i\in N} c_i$, and $\lim_{\lambda\to\infty} F(\lambda) = 0$. Thus for $E > 0$, there exists a unique $\lambda^* \geq 0$ such that $F(\lambda^*) = E$. For $E = 0$, then it is possible that there exists λ' and λ'' such that $F(\lambda') = F(\lambda'') = 0$. However, both λ' and λ'' would assign the same award, namely 0 for everyone.

where λ (the common tax imposed on everyone) is chosen so that $\sum_i CEL_i(N,c,E) = E$. This is commonly referred to as the *head tax*. The constrained equal loss rule is an equal sacrifice rule where $U_i(x) = U_j(x) = x$ for every $i, j \in \mathbb{N}$.

One intriguing equal sacrifice rule is the following. For every $i \in \mathbb{N}$, set

$$U_i(x) = -(w_i + x)^{-r_i},$$

for $w_i > 0$ and $r_i > 0$. In this instance, we can think of w_i as representing agent i's wealth level and r_i as representing her measure of relative risk aversion. This rule would be, in general, asymmetric and constrained.

2.2 Axioms

Our first three axioms are standard in the literature.

Continuity. For every problem (N,c,E), for every sequence of problems $\{(N,c^m,E^m)\}$, if $(N,c^m,E^m) \to (N,c,E)$ then $S(N,c^m,E^m) \to S(N,c,E)$.

Continuity simply requires that the rule be jointly continuous in total post-tax income and the vector of pre-tax incomes.

Consistency. For every problem (N,c,E), if $N' \subset N$ and $x = S(N,c,E)$, then $x_{N'} = S(N', c_{N'}, \sum_{N'} x_j)$.

Consistency imposes a restriction on the rule when the group shrinks. It says that how a rule assigns post-tax income among a subpopulation should not change when considered as a separate problem, fixing the total amount of post-tax income for that subpopulation.

Composition Down. For every problem (N,c,E), if $E' \in (E, \sum_N c_j]$ and $S(N,c,E') > \mathbf{0}$, then $S(N,c,E) = S(N, S(N,c,E'), E)$.

Imagine a scenario in which total post-tax income was determined to be E'. Citizens subsequently pay their respective assigned tax, leaving the post-tax allocation $S(N,c,E')$. Then it is discovered that the requisite tax revenue is larger than initially determined, decreasing total post-tax income to E. Composition Down says that the new post-tax income allocation can be determined either by using everyone's original income (i.e. c) or their previous post-tax income (i.e. $S(N,c,E')$); both methods will yield the same result. After all, if the rule S deemed $S(N,c,E')$ to be a fair way to allocate post-tax income under E', then it is reasonable to consider $S(N,c,E')$ as having all the necessary information to allocate post-tax income under E.

Our final axiom is, to our knowledge, new to the literature.

Lower Constrained Strict Claims Monotonicity (LCSM-Claims). For every problem (N,c,E) and $i \in N$, if $c_i' > c_i$ and $S(N,c,E) > \mathbf{0}$, then $S_i(N,(c_i', c_{-i}), E) > S_i(N,c,E)$.

LCSM-Claims says that as long as everyone has positive post-tax income, then if one agent's pre-tax income increases, that agent's post-tax income must increase. LCSM-Claims is similar to the standard Claims Monotonicity axiom and its strict version.

Claims Monotonicity. For every problem (N, c, E) and $i \in N$, if $c'_i > c_i$, then $S_i(N, (c'_i, c_{-i}), E) \geq S_i(N, c, E)$.

Strict Claims Monotonicity. For every problem (N, c, E) and $i \in N$, if $c'_i > c_i$, then $S_i(N, (c'_i, c_{-i}), E) > S_i(N, c, E)$.

Obviously Strict Claims Monotonicity implies LCSM-Claims. By itself, LCSM-Claims does not imply Claims Monotonicity. However when coupled with Continuity and Consistency, then LCSM-Claims does imply Claims Monotonicity.[4]

Lemma 1. *If S satisfies Continuity, Consistency, and LCSM-Claims, then S satisfies Claims Monotonicity.*

Proof. By way of contradiction, suppose not. Thus there exists (N, c, E), $i \in N$, and $c'_i > c_i$, such that $S_i(N, (c'_i, c_{-i}), E) < S_i(N, c, E)$. Set $x = S(N, c, E)$ and $x' = S(N, (c'_i, c_{-i}), E)$. Since $x_i > x'_i$, there must exist $j \in N$ such that $x'_j > x_j$. Consistency implies $(x_i, x_j) = S(\{i, j\}, (c_i, c_j), E)$ and $(x'_i, x'_j) = S(\{i, j\}, (c'_i, c_j), E)$. Obviously $x_i > 0$ and $x'_j > 0$. However, because of Continuity, we can assume $x'_i > 0$ and $x_j > 0$ without loss of generality. Hence we have $x_i > x'_i > 0$ and $x'_j > x_j > 0$. But this violates LCSM-Claims. □

Necessary But Not Sufficient

We conclude this section with a brief discussion of some axioms that are necessary, but not sufficient, for an equal sacrifice rule. It is trivial to show that Composition Down implies monotonicity in the total post-tax income.

Resource Monotonicity. For every problem (N, c, E), if $E' \in (E, \sum_N c_j]$, then $S(N, c, E') \geq S(N, c, E)$.

Lemma 2. *If S satisfies Composition Down, then S satisfies Resource Monotonicity.*

However, equal sacrifice rules satisfy an even stronger axiom than Resource Monotonicity.

Lower Constrained Strict Resource Monotonicity (LCSM-Resource). For every problem (N, c, E), if $E' \in (E, \sum_N c_j]$ and $S(N, c, E) > \mathbf{0}$, then $S(N, c, E') > S(N, c, E)$.

LCSM-Resource is similar in nature to LCSM-Claims. It says that as long as everyone has positive post-tax income, then increases in total post-tax income (i.e. decreases in the tax burden) will benefit all agents. It is not required that we include this axiom in our main result as it is implied by our other axioms.

[4]Claims Monotonicity is also implied by Composition Down and Consistency.

Lemma 3. *If S satisfies Consistency, Composition Down, and LCSM-Claims, then S satisfies LCSM-Resource.*

Proof. By way of contradiction, suppose S does not satisfy LCSM-Resource. I.e. there exists (N, c, E), $E' \in (E, \sum_N c_j]$, and $i \in N$ such that $S(N, c, E) > \mathbf{0}$ and $x_i \equiv S_i(N, c, E) \geq S_i(N, c, E')$. By Lemma 2, we must have $x_i = S_i(N, c, E')$. Since $E' > E$, there exists $j \in N$ such that $x_j \equiv S_j(N, c, E) < S_j(N, c, E') \equiv x'_j$. By Consistency, $(x_i, x_j) = S(\{i, j\}, (c_i, c_j), x_i + x_j)$ and $(x_i, x'_j) = S(\{i, j\}, (c_i, c_j), x_i + x'_j)$. By Composition Down, $x_j = S_j(\{i, j\}, (x_i, x'_j), x_i + x_j)$. However, this implies a violation of LCSM-Claims since $x_j = S_j(\{i, j\}, (x_i, x_j), x_i + x_j)$. □

3 Results

We now state our main results. Proofs for the following theorems are given in the appendix.

Theorem 1. *The rule S satisfies Continuity, Consistency, Composition Down, and LCSM-Claims if and only if $S \in \mathcal{ES}$.*

The following examples illustrate the extent to which the listed axioms are independent. For each axiom below, we give a rule which violates that axiom but satisfies the others in Theorem 1.

- **Consistency.** A rule that divides according to the flat tax for all two-person groups and according to the head tax for all groups larger than two.

- **Composition Down.** The symmetric parametric rule, S, with the parametric function[5]

$$f(c_0, \lambda) = \begin{cases} \frac{c_0}{1-\lambda c_0} & \text{if } \lambda < -\frac{1}{c_0}, \\ \frac{c_0}{2} & \text{if } -\frac{1}{c_0} \leq \lambda \leq \frac{1}{c_0}, \\ c_0 - \frac{c_0}{1+\lambda c_0} & \text{if } \lambda > \frac{1}{c_0}. \end{cases}$$

  Because S is a parametric rule, it satisfies Continuity and Consistency. Also, because f is strictly increasing in c_0, S must satisfy Strict Claims Monotonicity, which in turn implies that S must satisfy LCSM-Claims. However, S does not satisfy Composition Down. To see this, note that $S(\{1, 2\}, (6, 2), 4) = (3, 1)$, yet

$$\left(\frac{3}{2}, \frac{1}{2}\right) = S(\{1, 2\}, (3, 1), 2) \neq S(\{1, 2\}, (6, 2), 2) = \left(\frac{6}{2+\sqrt{10}}, \frac{6}{4+\sqrt{10}}\right).$$

- **LCSM-Claims.** The *leveling tax*: This tax assigns the same post-tax income to all agents, with the proviso that no agent's post-tax income is more than their respective pre-tax income. This is also called the *constrained equal awards rule* in the conflicting claims literature.

[5]The family of symmetric parametric rules is characterized in Young (1987). See section 4 for a discussion of this family.

It is an open question whether Continuity is independent of the other axioms. However, we note that it is easy to show that Composition Down implies continuity in the total post-tax income E. Thus the only question is whether continuity in the pre-tax income vector c is implied by the other axioms.

An important question regarding the equal sacrifice rules is: To what extent is an equal sacrifice representation unique? This is answered by the next theorem.

Theorem 2. *Suppose $S \in \mathcal{ES}$ has two representations: $U, V \in \mathcal{U}$. Then there exist $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}^{\mathbb{N}}$ such that $V_i = \alpha U_i + \beta_i$ for every i.*

Thus only certain manipulations of the equal sacrifice representation are allowed without changing the underlying rule. Namely, the individual utility functions may be rescaled by a common factor. Also, each utility function may be shifted by any amount.

A special case of the family of equal sacrifice rules are those that are symmetric in the sense that every individual is assigned the same utility function. I.e. $U_i = U_j$ for all $i, j \in \mathbb{N}$. Let $\mathcal{ES}^*$ denote the family of symmetric equal sacrifice rules. The following axiom is implied by the family of symmetric equal sacrifice rules.

Symmetry. For every problem (N, c, E) and $i, j \in N$, if $c_i = c_j$, then $S_i(N, c, E) = S_j(N, c, E)$.

This axiom imposes the requirement that individuals with the same pre-tax income will have the same post-tax income (which implies the same tax for these individuals). In conjunction with our other axioms, Symmetry is also sufficient to guarantee symmetric equal sacrifice rules.

Theorem 3. *The rule S satisfies Continuity, Consistency, Composition Down, LCSM-Claims, and Symmetry if and only if $S \in \mathcal{ES}^*$.*

We conclude with a result that examines what is needed to guarantee that the equal sacrifice representation of a rule is concave.

Linked Claims-Resource Monotonicity. For every problem (N, c, E) and $i \in N$, if $h > 0$, then $h \geq S_i(N, (c_i + h, c_{-i}), E + h) - S_i(N, c, E)$.

In the context of taxation, Linked Claims-Resource Monotonicity says that if an agent's pre-tax income increases, then her tax burden must weakly increase. It is easy to show that Linked Claims-Resource Monotonicity is the dual to Claims Monotonicity.[6] This is enough to guarantee concave utility functions.

Theorem 4. *The rule ES^U satisfies Linked Claims-Resource Monotonicity if and only if U_i is concave for every i.*

[6]See Thomson and Yeh (2008) for further discussion of this result, as well as the duality operator in general.

4 Related Literature

The family of equal sacrifice rules is a special case of the family of parametric rules characterized by Stovall (2014a). A parametric rule is defined by a continuous function $f : \mathbb{N} \times \mathbb{R}_{++} \times [a, b] \to \mathbb{R}_+$, where $-\infty \leq a < b \leq \infty$, such that (i) f is weakly increasing in the third argument, and (ii) for every $i \in \mathbb{N}$ and $c_0 \in \mathbb{R}_{++}$ we have $f(i, c_0, a) = 0$ and $f(i, c_0, b) = c_0$. A division rule Par^f is defined as follows. For every (N, c, E) and for every $i \in N$,

$$Par_i^f(N, c, E) \equiv f(i, c_i, \lambda),$$

where λ is chosen so that $\sum_N f(i, c_i, \lambda) = E$.[7] The axioms that characterize the family of parametric rules are Continuity, Consistency, Resource Monotonicity, as well as two other technical axioms referred to as Intrapersonal Consistency and N-Continuity. Let $\mathcal{P}$ denote the family of parametric rules.

An important special case of this family is the family of symmetric parametric rules, originally characterized by Young (1987). The axioms that characterize the family of symmetric parametric rules are Continuity, Consistency, and Symmetry. Let $\mathcal{P}^*$ denote the family of symmetric parametric rules.

It is easy to see that $\mathcal{ES} \subset \mathcal{P}$. Let $a = -\infty$ and $b = 0$. Then for $U \in \mathcal{U}$ and for every $i \in \mathbb{N}$, define the parametric function

$$f(i, c_i, \lambda) \equiv \overline{U_i^{-1}}\left(U_i(c_i) + \lambda\right).$$

(Similarly, we have $\mathcal{ES}^* \subset \mathcal{P}^*$.) Thus it must be (though we do not show this directly) that the set of axioms used in Theorem 1 imply the two technical axioms used by Stovall (2014a), Intrapersonal Consistency and N-Continuity.

We turn now to the papers most closely related to the present work. To aid in the discussion, we introduce some notation. Let $\widehat{\mathcal{ES}}$ denote the family of unconstrained equal sacrifice rules. I.e. $ES^U \in \widehat{\mathcal{ES}}$ if $U \in \mathcal{U}$ satisfies $\lim_{x\to 0} U_i(x) = -\infty$ for all $i \in \mathbb{N}$. Set $\widehat{\mathcal{ES}}^* \equiv \mathcal{ES}^* \cap \widehat{\mathcal{ES}}$, which is the family of symmetric unconstrained equal sacrifice rules. (Thus an asterisk denotes a family that is symmetric, while a hat denotes a family that is unconstrained.)

Young (1988) provides a characterization of $\widehat{\mathcal{ES}}^*$. The axioms he used are Continuity, Consistency, Composition Down, Symmetry, and two other axioms not yet introduced called Strict Resource Monotonicity and Strict Order Preservation of Awards.

Strict Resource Monotonicity. For every problem (N, c, E), if $E' \in (E, \sum_N c_j]$, then $S(N, c, E') > S(N, c, E)$.

Strict Order Preservation of Awards. For every problem (N, c, E), if $i, j \in N$, $c_i > c_j$, and $E > 0$, then $S_i(N, c, E) > S_j(N, c, E)$.

Lemma 4. *Suppose S satisfies Continuity, Consistency, and Symmetry. Then S satisfies Strict Order Preservation of Awards if and only if S satisfies Strict Claims Monotonicity.*

[7] This rule is well-defined because such a λ always exists, and if there are multiple such lambdas, the underlying allocation is the same for them.

Proof. Applying Young (1987, Theorem 1), S is a symmetric parametric rule. Let f denote the parametric representation of S. It is not hard to see that S satisfies Strict Order Preservation of Awards if and only if for any λ, $f(c_0, \lambda)$ is strictly increasing in c_0. Similarly, S satisfies Strict Claims Monotonicity if and only if for any λ, $f(c_0, \lambda)$ is strictly increasing in c_0. Thus S satisfies Strict Order Preservation of Awards if and only if S satisfies Strict Claims Monotonicity. □

We can easily modify the proof of Lemma 3 to get the following result.

Lemma 5. *If S satisfies Consistency, Composition Down, and Strict Claims Monotonicity, then S satisfies Strict Resource Monotonicity.*

Together, Lemma 4 and Lemma 5 imply the following alternate (and slightly tighter) characterization of the family $\widehat{\mathcal{ES}}^*$.

Theorem 5 (Alternate to Young (1988, Theorem 1))**.** *The rule S satisfies Continuity, Consistency, Composition Down, Strict Claims Monotonicity, and Symmetry if and only if $S \in \widehat{\mathcal{ES}}^*$.*

Chambers and Moreno-Ternero (2017) characterize a family of rules which contains (but is broader than) $\mathcal{ES}^*$. Combining their nomenclature with ours, we will refer to the family they characterize as the generalized symmetric equal sacrifice rules, denoted $\mathcal{GES}^*$. The axioms that characterize this family of rules are Continuity, Consistency, Composition Down, and Symmetry. Given this characterization, it is easy to see that $\mathcal{ES}^* \subset \mathcal{GES}^* \subset \mathcal{P}^*$.

Naumova (2002) provides a characterization of $\widehat{\mathcal{ES}}$. However, her definition of a problem is broader than the one we consider, allowing for the possibility of surplus sharing.[8] Because of this, her main axiom, called Path Independence, is stronger than Composition Down. Not only this, but Path Independence implies Strict Claims Monotonicity, though this is not explicitly shown by Naumova. Also, Strict Resource Monotonicity is explicitly imposed. However, given Lemma 5 and Theorem 1, the following alternate characterization of $\widehat{\mathcal{ES}}$ can easily be proven.

Theorem 6 (Alternate to Naumova (2002, Theorem 2.1))**.** *The rule S satisfies Continuity, Consistency, Composition Down, and Strict Claims Monotonicity if and only if $S \in \widehat{\mathcal{ES}}$.*

Table 1 summarizes this discussion by listing the axioms each of the above families of rules respectively satisfy. Figure 1 illustrates the logical relationships between these families. Returning to the examples given at the end of subsection 2.1, the proportional rule is a member of $\widehat{\mathcal{ES}}^*$ and the constrained equal loss rule is a member of $\mathcal{ES}^*$. The third example given is a member of $\mathcal{ES}$.

We conclude with the following observation. As pointed out by Young (1988, p.322), there are other ways to interpret the principle of equal sacrifice. For example, one may wish to instead equalize *marginal* sacrifice across individuals. However, since

[8]A surplus sharing problem is similar to a conflicting claims problem, but where $E \geq \sum_i c_i$.

	$\mathcal{P}$	$\mathcal{P}^*$	$\mathcal{ES}$	$\mathcal{GES}^*$	$\mathcal{ES}^*$	$\widehat{\mathcal{ES}}$	$\widehat{\mathcal{ES}}^*$
Continuity	⊕	⊕	⊕	⊕	⊕	⊕	⊕
N-Continuity	⊕	+	+	+	+	+	+
Consistency	⊕	⊕	⊕	⊕	⊕	⊕	⊕
Intrapersonal Consistency	⊕	+	+	+	+	+	+
Symmetry	−	⊕	−	⊕	⊕	−	⊕
Composition Down	−	−	⊕	⊕	⊕	⊕	⊕
Claims Monotonicity	−	−	+	+	+	+	+
LCSM-Claims	−	−	⊕	−	⊕	+	+
Strict Claims Monotonicity	−	−	−	−	−	⊕	⊕
Resource Monotonicity	⊕	+	+	+	+	+	+
LCSM-Resource	−	−	+	−	+	+	+
Strict Resource Monotonicity	−	−	−	−	−	+	+
Source	[1]	[2]	[3]	[4]	[5]	[6]	[7]

[1] Stovall (2014a, Theorem 1).
[2] Young (1987, Theorem 1).
[3] Theorem 1.
[4] Chambers and Moreno-Ternero (2017, Theorem 1).
[5] Theorem 3.
[6] Theorem 6, which is an alternate characterization of the family studied by Naumova (2002, Theorem 2.1).
[7] Theorem 5, which is a tighter result based off Young (1988, Theorem 1).

Table 1: Summary of families of rules and axioms. *The symbols + and − indicate the axiom is necessary and not necessary, respectively. For any column, the set of axioms indicated by ⊕ are necessary and sufficient for the given family.*

the marginal sacrifice of another dollar of taxation is identical to the marginal utility of another dollar of income, this equates to simply choosing post-tax income so as to maximize the sum of utilities. This is exactly the method of rules characterized by Stovall (2014b).

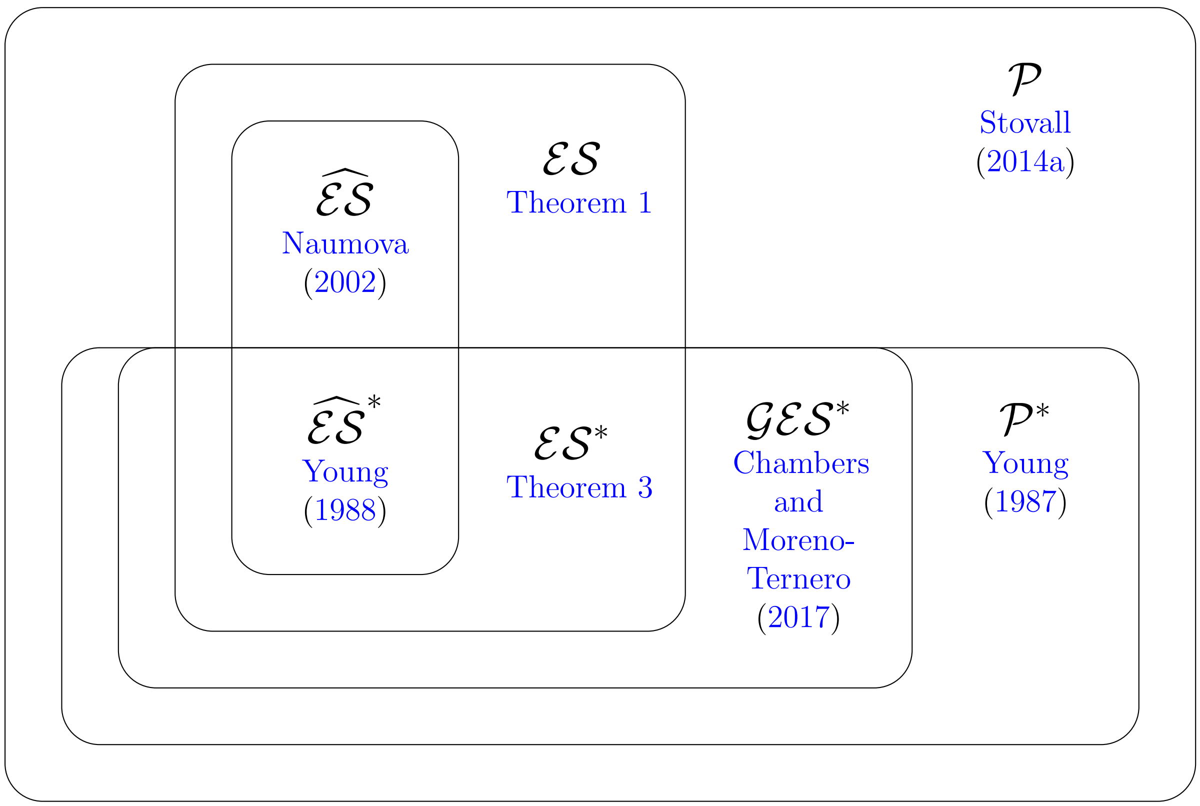

Figure 1: Diagram of logical relation among families of rules.

Appendix

A Notation

For any set A, let A° denote the interior of A. Let $\mathbb{D}$ denote the set of (non-negative) dyadic rationals, i.e.

$$\mathbb{D} \equiv \left\{ \frac{m}{2^n} : m, n \in \mathbb{N} \cup \{0\} \right\}.$$

For an interval $[a, b]$ and $\mathbb{K} = \mathbb{N}, \mathbb{D}$, etc. let $\mathbb{K}[a, b]$ denote the set $\mathbb{K} \cap [a, b]$. Similar notation will be used for open or unbounded intervals.

Definition. For $a < b$ and $M \in \mathbb{N}$, we say $\{y^m\}_{m=0}^{M}$ is an M*-partition* of $[a, b]$ if

$$a = y^0 < y^1 < \ldots < y^{M-1} < y^M = b.$$

B Proof of Theorem 1

Proving that the axioms are necessary is a straightforward exercise. Thus we only show that the axioms are sufficient to yield an equal sacrifice representation. So, let S satisfy Continuity, Consistency, Composition Down, and LCSM-Claims. By Lemma 3, S also satisfies LCSM-Resource.

B.1 Definitions and Preliminary Results

Define

$$Y \equiv \{(i, c_i, x_i) : i \in \mathbb{N}, 0 < x_i \leq c_i\}.$$

We call $(i, c_i, x_i) \in Y$ a *situation*. We think of a situation (i, c_i, x_i) as describing an agent i, her pre-tax income c_i, and her post-tax income x_i. Define the binary relation $\succsim$ over Y:

$$(i, c_i, x_i) \succsim (j, c_j, x_j) \text{ if } S_i(\{i, j\}, (c_i, c_j), x_i + x_j) \geq x_i.$$

Let $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$ respectively. Note that $(i, c_i, x_i) \sim (j, c_j, x_j)$ if and only if $S(\{i, j\}, (c_i, c_j), x_i + x_j) = (x_i, x_j)$. In fact, Consistency implies the following result.

Lemma 6. *Suppose $x = S(N, c, E)$. Then for every $i, j \in N$ such that $x_i, x_j > 0$, we have $(i, c_i, x_i) \sim (j, c_j, x_j)$.*

The next two lemmas will be invoked often. We omit their proofs as they follow easily from Continuity, LCSM-Resource, and LCSM-Claims.

Lemma 7. *Suppose $(i, c_i, x_i) \succ (j, c_j, x_j)$. Then $x_i < c_i$ and there exists a unique $x_i' \in (x_i, c_i]$ such that $(i, c_i, x_i') \sim (j, c_j, x_j)$. Moreover, $x_i' = c_i$ if and only if $x_j = c_j$.*

Lemma 8. *Suppose $(i, c_i, x_i) \sim (j, c_j, x_j)$. Then*

(i) $(i, c_i, x_i') \succ (j, c_j, x_j)$ *for every* $x_i' \in (0, x_i)$;

(ii) $(i, c_i, x_i) \succ (j, c_j, x'_j)$ *for every* $x'_j \in (x_j, c_j]$*; and*

(iii) $(i, c_i, x_i) \succ (j, c'_j, x_j)$ *for every* $c'_j \in [x_j, c_j)$.

Lemma 9. *Suppose* $(i, c_i, x_i) \sim (j, c_j, x_j)$.

(i) For every $x'_j \in [x_j, c_j]$*, there exists a unique* $\hat{x}_i(x'_j) \in [x_i, c_i]$ *such that* $(i, c_i, \hat{x}_i(x'_j)) \sim (j, c_j, x'_j)$*. Moreover,* $\hat{x}_i(x'_j)$ *is continuous and strictly increasing, with* $\hat{x}_i(x_j) = x_i$ *and* $\hat{x}_i(c_j) = c_i$.

(ii) For every $c'_j \in [x_j, c_j]$*, there exists a unique* $\hat{x}_i(c'_j) \in [x_i, c_i]$ *such that* $(i, c_i, \hat{x}_i(c'_j)) \sim (j, c'_j, x_j)$*. Moreover,* $\hat{x}_i(c'_j)$ *is continuous and strictly decreasing, with* $\hat{x}_i(x_j) = c_i$ *and* $\hat{x}_i(c_j) = x_i$.

Proof. (i) The existence of $\hat{x}_i(x'_j)$ follows easily from Lemma 7 and item (ii) of Lemma 8. LCSM-Resource and Continuity imply that $\hat{x}_i(x'_j)$ is strictly increasing and continuous.

(ii) The existence of $\hat{x}_i(c'_j)$ follows easily from Lemma 7 and item (iii) of Lemma 8. LCSM-Claims and Continuity imply that $\hat{x}_i(c'_j)$ is strictly decreasing and continuous. □

The next two lemmas establish that $\sim$ is transitive.

Lemma 10. *Suppose* $(i, c_i, x_i) \sim (j, c_j, x_j) \sim (k, c_k, x_k)$*, where* $i \neq k$*. Then there exists* E *such that*

$$S(\{i, j, k\}, (c_i, c_j, c_k), E) = (x_i, x_j, x_k).$$

Proof. By Continuity, there exists E such that $S_i(\{i, j, k\}, (c_i, c_j, c_k), E) = x_i$. Let $x'_j = S_j(\{i, j, k\}, (c_i, c_j, c_k), E)$ and $x'_k = S_k(\{i, j, k\}, (c_i, c_j, c_k), E)$. Consistency then implies $(x_i, x_j) = S(\{i, j\}, (c_i, c_j), x_i + x'_j)$, or $(i, c_i, x_i) \sim (j, c_j, x'_j)$. Since $(i, c_i, x_i) \sim (j, c_j, x_j)$, item (i) of Lemma 9 then implies $x'_j = x_j$. Similarly, we can show $x'_k = x_k$. □

Lemma 11. *Suppose* $(i, c_i, x_i) \sim (j, c_j, x_j) \sim (k, c_k, x_k)$*, where* $i \neq k$*. Then* $(i, c_i, x_i) \sim (k, c_k, x_k)$.

Proof. This follows directly from Lemma 6 and Lemma 10. □

The final lemma in this subsection establishes the existence of what we think of as a 'halfway point' between a claims vector and its associated awards vector.

Lemma 12. *Let* (N, x^1, E) *be a problem where* $|N| \geq 3$*. Suppose* $x^0 = S(N, c, E) > \mathbf{0}$*. Then there exists a unique* $x^{1/2}$ *satisfying* $x^0 < x^{1/2} < x^1$ *such that for any* $i, j \in N$ *and* $m, m' \in \{1, 2\}$*, we have*

$$(i, x_i^{m/2}, x_i^{(m-1)/2}) \sim (j, x_j^{m'/2}, x_j^{(m'-1)/2}).$$

Proof. Fix $i, j \in N$. By Lemma 6, $(i, x_i^1, x_i^0) \sim (j, x_j^1, x_j^0)$. By item (i) of Lemma 9, there exists $\hat{x}_i(\cdot)$ continuous and strictly increasing such that for any $a \in [x_j^0, x_j^1]$, we have $(i, x_i^1, \hat{x}_i(a)) \sim (j, x_j^1, a)$. Note that when $a = x_j^0$, then $\hat{x}_i(a) = x_i^0$, so $(j, x_j^1, a) \succ (i, \hat{x}_i(a), x_i^0)$ by item (iii) of Lemma 8. Also, when $a = x_j^1$, then $\hat{x}_i(a) = x_i^1$, so $(i, \hat{x}_i(a), x_i^0) \succ (j, x_j^1, a)$ by part (ii) of Lemma 8.

Because S and $\hat{x}_i$ are continuous, there exists $x_j^{1/2} \in (x_j^0, x_j^1)$ such that $(j, x_j^1, x_j^{1/2}) \sim (i, \hat{x}_i(x_j^{1/2}), x_i^0)$. Set $x_i^{1/2} = \hat{x}_i(x_j^{1/2}) \in (x_i^0, x_i^1)$. Thus $(i, x_i^1, x_i^{1/2}) \sim (j, x_j^1, x_j^{1/2}) \sim (i, x_i^{1/2}, x_i^0)$. Since $(i, x_i^1, x_i^0) \sim (j, x_j^1, x_j^0)$ and $(i, x_i^1, x_i^{1/2}) \sim (j, x_j^1, x_j^{1/2})$, Composition Down implies $(i, x_i^{1/2}, x_i^0) \sim (j, x_j^{1/2}, x_j^0)$. Thus we have

$$(i, x_i^1, x_i^{1/2}) \sim (j, x_j^1, x_j^{1/2}) \sim (i, x_i^{1/2}, x_i^0) \sim (j, x_j^{1/2}, x_j^0).$$

Now fix $k \in N \setminus \{i, j\}$. By Lemma 6, $(k, x_k^1, x_k^0) \sim (j, x_j^1, x_j^0)$. By item (i) of Lemma 9, there exists $x_k^{1/2} = \hat{x}_k(x_j^{1/2}) \in (x_k^0, x_k^1)$ such that $(k, x_k^1, x_k^{1/2}) \sim (j, x_j^1, x_j^{1/2})$. Repeatedly applying Lemma 11 gives $(k, x_k^1, x_k^{1/2}) \sim (j, x_j^{1/2}, x_j^0)$ and $(k, x_k^1, x_k^{1/2}) \sim (i, x_i^1, x_i^{1/2})$. But then applying Lemma 11 one more time gives $(i, x_i^1, x_i^{1/2}) \sim (j, x_j^{1/2}, x_j^0)$. Thus we have

$$(i, x_i^1, x_i^{1/2}) \sim (j, x_j^1, x_j^{1/2}) \sim (i, x_i^{1/2}, x_i^0) \sim (j, x_j^{1/2}, x_j^0) \sim (i, x_i^1, x_i^{1/2}).$$

Indeed, similar reasoning yields

$$(k, x_k^1, x_k^{1/2}) \sim (j, x_j^1, x_j^{1/2}) \sim (k, x_k^{1/2}, x_k^0) \sim (j, x_j^{1/2}, x_j^0) \sim (k, x_k^1, x_k^{1/2}).$$

A similar process can show the above relations for any two agents in N. □

B.2 Measuring a Situation

In this subsection, we establish a way of measuring a situation. Roughly, this is done by arbitrarily choosing three situations that are equivalent under $\sim$ to be the unit. For each of these 'units', the dyadic set is defined by recursively applying Lemma 12. This will allow us to measure situations that are 'less' than the unit. Thus the measure of a given situation will be the number of times the unit 'covers' the given situation.

Fix $x^1 \in \mathbb{R}^3_{++}$. By LCSM-Resource, there exists $E \in (0, \sum_{i=1}^3 x_i^1)$ such that $x^0 \equiv S(\{1,2,3\}, x^1, E) > \mathbf{0}$.

For $i \in \{1,2,3\}$, define the function $x_i : \mathbb{D}[0,1] \to [x_i^0, x_i^1]$ recursively as follows.[9] Set $x_i(0) = x_i^0$, $x_i(1) = x_i^1$. For $n = 1, 2, \ldots$ and $m \in \mathbb{N}[1, 2^{n-1}]$, let $x_i\left(\frac{2m-1}{2^n}\right) \in \left(x_i\left(\frac{2m-2}{2^n}\right), x_i\left(\frac{2m}{2^n}\right)\right)$ denote the unique numbers from Lemma 12 satisfying

$$\left(i, x_i\left(\tfrac{2m-m'}{2^n}\right), x_i\left(\tfrac{2m-m'-1}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{2m-m''}{2^n}\right), x_j\left(\tfrac{2m-m''-1}{2^n}\right)\right) \tag{1}$$

for $j \in \{1,2,3\} \setminus i$ and $m', m'' \in \{0, 1\}$.

[9]The use of the group $\{1,2,3\}$ is arbitrary. What is necessary is having a group of at least three agents so as to take full advantage of the implications of Consistency.

Lemma 13. *For any $n \in \mathbb{N}[2,\infty)$, $m \in \mathbb{N}[2,2^{n-1}]$, and $i,j \in \{1,2,3\}$,*

$$\left(i, x_i\left(\tfrac{2m-1}{2^n}\right), x_i\left(\tfrac{2m-2}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{2m-2}{2^n}\right), x_j\left(\tfrac{2m-3}{2^n}\right)\right).$$

Proof. Fix $n \in \mathbb{N}[2,\infty)$ and $m \in \mathbb{N}[2,2^{n-1}]$. For any $i,j \in \{1,2,3\}$, (1) implies

$$\left(i, x_i\left(\tfrac{2m-1}{2^n}\right), x_i\left(\tfrac{2m-2}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{2m-1}{2^n}\right), x_j\left(\tfrac{2m-2}{2^n}\right)\right) \tag{2}$$

and

$$\left(i, x_i\left(\tfrac{2m-2}{2^n}\right), x_i\left(\tfrac{2m-3}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{2m-2}{2^n}\right), x_j\left(\tfrac{2m-3}{2^n}\right)\right). \tag{3}$$

Composition Down then implies

$$\left(i, x_i\left(\tfrac{2m-1}{2^n}\right), x_i\left(\tfrac{2m-3}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{2m-1}{2^n}\right), x_j\left(\tfrac{2m-3}{2^n}\right)\right).$$

Since this holds for all $i,j \in \{1,2,3\}$, Lemma 10 and Lemma 12 imply the existence of a unique half-point. But (2) and (3) then imply that this half-point must be $x_i(\frac{2m-2}{2^n})$ for $i \in \{1,2,3\}$. Thus Lemma 12 gives the desired result. □

Lemma 14. *For any $n \in \mathbb{N}$, $m, m' \in \mathbb{N}[1,2^n]$, and $i,j \in \{1,2,3\}$*

$$\left(i, x_i\left(\tfrac{m}{2^n}\right), x_i\left(\tfrac{m-1}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{m'}{2^n}\right), x_j\left(\tfrac{m'-1}{2^n}\right)\right).$$

Proof. For $n = 1$, the result is true by (1). So assume $n \geq 2$. Without loss of generality, assume $m > m'$.

Case 1: m is odd, i.e. $m = 2\hat{m} - 1$ for some $\hat{m} \in \mathbb{N}[2,2^{n-1}]$. Then by Lemma 13

$$\left(i, x_i\left(\tfrac{m}{2^n}\right), x_i\left(\tfrac{m-1}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{m-1}{2^n}\right), x_j\left(\tfrac{m-2}{2^n}\right)\right).$$

Equation (1) then implies

$$\left(j, x_j\left(\tfrac{m-1}{2^n}\right), x_j\left(\tfrac{m-2}{2^n}\right)\right) \sim \left(i, x_i\left(\tfrac{m-2}{2^n}\right), x_i\left(\tfrac{m-3}{2^n}\right)\right).$$

Repeatedly applying Lemma 13 and (1) yields a chain of relations $\sim$ from $\left(i, x_i\left(\frac{m}{2^n}\right), x_i\left(\frac{m-1}{2^n}\right)\right)$ to $\left(j, x_j\left(\frac{m'}{2^n}\right), x_j\left(\frac{m'-1}{2^n}\right)\right)$. Moreover, these relations hold for all $i,j \in \{1,2,3\}$. Repeated application of Lemma 11 then yields the desired result.

Case 2: m is even. The proof is similar to the first case, only applying (1) first and then Lemma 13 second. □

Lemma 15. *Let $d, d', \hat{d}, \hat{d}' \in \mathbb{D}[0,1]$ satisfy $d - d' = \hat{d} - \hat{d}' > 0$. Then for any $i,j \in \{1,2,3\}$, we have*

$$(i, x_i(d), x_i(d')) \sim (j, x_j(\hat{d}), x_j(\hat{d}')).$$

Proof. Since $d, d', \hat{d}, \hat{d}' \in \mathbb{D}[0,1]$ and $d - d' = \hat{d} - \hat{d}'$, there exists $n \in \mathbb{N}$, $m, \hat{m} \in \mathbb{N}[1,2^n]$, and $\bar{m} \in \mathbb{N}[1, \min\{m, \hat{m}\}]$ such that $d = \frac{m}{2^n}$, $d' = \frac{m-\bar{m}}{2^n}$, $\hat{d} = \frac{\hat{m}}{2^n}$, and $\hat{d}' = \frac{\hat{m}-\bar{m}}{2^n}$.

By Lemma 14, we have

$$\left(i, x_i\left(\tfrac{m}{2^n}\right), x_i\left(\tfrac{m-1}{2^n}\right)\right) \sim \left(j, x_j\left(\tfrac{\hat{m}}{2^n}\right), x_j\left(\tfrac{\hat{m}-1}{2^n}\right)\right)$$

for any $i, j \in \{1, 2, 3\}$. Similarly, we have

$$\left(i, x_i\left(\frac{m-1}{2^n}\right), x_i\left(\frac{m-2}{2^n}\right)\right) \sim \left(j, x_j\left(\frac{\hat{m}-1}{2^n}\right), x_j\left(\frac{\hat{m}-2}{2^n}\right)\right).$$

Composition Down then implies

$$\left(i, x_i\left(\frac{m}{2^n}\right), x_i\left(\frac{m-2}{2^n}\right)\right) \sim \left(j, x_j\left(\frac{\hat{m}}{2^n}\right), x_j\left(\frac{\hat{m}-2}{2^n}\right)\right).$$

Continuing in this way, we have

$$\left(i, x_i\left(\frac{m}{2^n}\right), x_i\left(\frac{m-\bar{m}}{2^n}\right)\right) \sim \left(j, x_j\left(\frac{\hat{m}}{2^n}\right), x_j\left(\frac{\hat{m}-\bar{m}}{2^n}\right)\right),$$

or

$$(i, x_i(d), x_i(d')) \sim (j, x_j(\hat{d}), x_j(\hat{d}')),$$

as desired. □

Now for $i \in \{1, 2, 3\}$, we extend x_i from $\mathbb{D}[0, 1]$ to $[0, 1]$: For $a \in [0, 1]$, set

$$x_i(a) = \sup\{x_i(d) : d \in \mathbb{D}[0, 1] \text{ and } d \leq a\}.$$

The density of $\mathbb{D}$ in $\mathbb{R}$ in conjunction with Continuity and LCSM-Resource imply the following.

Lemma 16. *For $i \in \{1, 2, 3\}$, the function $x_i : [0, 1] \to [x_i^0, x_i^1]$ is continuous and strictly increasing.*

The following lemma follows easily from Lemma 15.

Lemma 17. *Let $a, b, \hat{a}, \hat{b} \in [0, 1]$ satisfy $b - a' = \hat{b} - \hat{a} > 0$. Then for any $i, j \in \{1, 2, 3\}$, we have*

$$(i, x_i(b), x_i(a)) \sim (j, x_j(\hat{b}), x_j(\hat{a})).$$

For $i \in \{1, 2, 3\}$, let u_i denote the inverse of x_i. I.e. for $x \in [x_i^0, x_i^1]$, we have $u_i(x) = a$ if $x_i(a) = x$. Lemma 16 then implies the following lemma.

Lemma 18. *For $i \in \{1, 2, 3\}$, the function $u_i : [x_i^0, x_i^1] \to [0, 1]$ is continuous and strictly increasing.*

Lemma 19. *Fix $i, j \in \{1, 2, 3\}$, $\hat{x}_i, \hat{x}'_i \in [x_i(0), x_i(1)]$, and $\hat{x}_j, \hat{x}'_j \in [x_j(0), x_j(1)]$. Then $(i, \hat{x}_i, \hat{x}'_i) \sim (j, \hat{x}_j, \hat{x}'_j)$ if and only if $u_i(\hat{x}_i) - u_i(\hat{x}'_i) = u_j(\hat{x}_j) - u_j(\hat{x}'_j)$.*

Proof. ($\Rightarrow$) By way of contradiction and without loss of generality, suppose $u_i(\hat{x}_i) - u_i(\hat{x}'_i) > u_j(\hat{x}_j) - u_j(\hat{x}'_j)$. Since u_i is continuous and strictly increasing by Lemma 18, there exists a unique $\hat{x}''_i \in (\hat{x}'_i, \hat{x}_i)$ such that $u_i(\hat{x}_i) - u_i(\hat{x}''_i) = u_j(\hat{x}_j) - u_j(\hat{x}'_j)$. Lemma 17 then implies

$$(i, x_i(u_i(\hat{x}_i)), x_i(u_i(\hat{x}''_i))) \sim (j, x_j(u_j(\hat{x}_j)), x_j(u_j(\hat{x}'_j))),$$

or

$$(i, \hat{x}_i, \hat{x}''_i) \sim (j, \hat{x}_j, \hat{x}'_j).$$

But since $\hat{x}''_i > \hat{x}'_i$, item (i) of Lemma 8 implies $(i, \hat{x}_i, \hat{x}'_i) \succ (j, \hat{x}_j, \hat{x}'_j)$, which is a contradiction.

($\Leftarrow$) This direction is a direct result of Lemma 17. □

Lemma 20. *Fix $i \in \{1,2,3\}$. Let $\hat{x}_i, \hat{x}'_i, \bar{x}_i, \bar{x}'_i \in [x_i(0), x_i(1)]$ and $(j, c_j, x_j) \in Y$ satisfy $j \neq i$ and*

$$(i, \hat{x}_i, \hat{x}'_i) \sim (j, c_j, x_j) \sim (i, \bar{x}_i, \bar{x}'_i).$$

Then $u_i(\hat{x}_i) - u_i(\hat{x}'_i) = u_i(\bar{x}_i) - u_i(\bar{x}'_i)$.

Proof. Choose $k \in \{1,2,3\} \setminus \{i,j\}$. Since u_k is strictly increasing and continuous by Lemma 18, there exists a unique $\hat{x}_k \in [x_k(0), x_k(1)]$ such that $u_k(\hat{x}_k) = u_i(\hat{x}_i) - u_i(\hat{x}'_i)$. Since $u_k(x_k(0)) = 0$, this means $u_k(\hat{x}_k) - u_k(x_k(0)) = u_i(\hat{x}_i) - u_i(\hat{x}'_i)$, so by Lemma 19 we have $(k, \hat{x}_k, x_k(0)) \sim (i, \hat{x}_i, \hat{x}'_i)$. Lemma 11 applied twice implies $(k, \hat{x}_k, x_k(0)) \sim (i, \bar{x}_i, \bar{x}'_i)$. Lemma 19 implies $u_k(\hat{x}_k) - u_k(x_k(0)) = u_i(\bar{x}_i) - u_i(\bar{x}'_i)$. Thus $u_i(\hat{x}_i) - u_i(\hat{x}'_i) = u_i(\bar{x}_i) - u_i(\bar{x}'_i)$. □

The next lemma will be a key part in establishing a measure of a situation.

Lemma 21. *For any $(i, c_i, x_i) \in Y^{\mathrm{o}}$ and $j \in \{1,2,3\} \setminus \{i\}$, there exist unique*

(i) $\{y^m\}_{m=0}^M$ an M-partition of $[x_i, c_i]$, and

(ii) $\ell \in [x_j(0), x_j(1))$

such that $(i, y^m, y^{m-1}) \sim (j, x_j(1), x_j(0))$ for every $m \in \{2, \ldots, M\}$, and $(i, y^1, y^0) \sim (j, x_j(1), \ell)$.

Proof. Fix $(i, c_i, x_i) \in Y^{\mathrm{o}}$ and $j \in \{1,2,3\} \setminus \{i\}$. Define the sequence $\{\hat{y}^m\}$ recursively: Set $\hat{y}^0 = c_i$. If $(i, \hat{y}^{m-1}, x_i) \succ (j, x_j(1), x_j(0))$, then by Lemma 7 there exists a unique $\bar{y} \in (x_i, \hat{y}^{m-1})$ such that $(i, \hat{y}^{m-1}, \bar{y}) \sim (j, x_j(1), x_j(0))$. (Note that $\bar{y} \neq \hat{y}^{m-1}$ since $x_1(1) \neq x_1(0)$.) Set $\hat{y}^m = \bar{y}$ so that $(i, \hat{y}^{m-1}, \hat{y}^m) \sim (j, x_j(1), x_j(0))$. If $(j, x_j(1), x_j(0)) \succsim (i, \hat{y}^{m-1}, x_i)$, then set $\hat{y}^m = x_i$ and $M = m$. The following claim shows that this case will happen for finite m.

Claim: There exists m such that $(j, x_j(1), x_j(0)) \succsim (i, \hat{y}^{m-1}, x_i)$. By way of contradiction, suppose not, i.e. $(i, \hat{y}^m, x_i) \succ (j, x_j(1), x_j(0))$ for all $m \in \mathbb{N}$. Since $\{\hat{y}^m\}_{\mathbb{N}}$ is a strictly decreasing sequence with a lower bound x_i, it converges. Let $\hat{y}^m \to \tilde{y} \geq x_i$. Consider the sequence of problems $\{(\{i,j\}, (\hat{y}^{m-1}, x_j(1)), \hat{y}^m + x_j(0))\}_{\mathbb{N}}$. Since $(i, \hat{y}^{m-1}, \hat{y}^m) \sim (j, x_j(1), x_j(0))$ for every $m \in \mathbb{N}$, this means $S(\{i,j\}, (\hat{y}^{m-1}, x_j(1)), \hat{y}^m + x_j(0)) = (\hat{y}^m, x_j(0))$ for every $m \in \mathbb{N}$. Thus

$$S(\{i,j\}, (\hat{y}^{m-1}, x_j(1)), \hat{y}^m + x_j(0)) \to (\tilde{y}, x_j(0)).$$

However Continuity implies

$$S(\{i,j\}, (\hat{y}^{m-1}, x_j(1)), \hat{y}^m + x_j(0)) \to S(\{i,j\}, (\tilde{y}, x_j(1)), \tilde{y} + x_j(0)).$$

Thus $S(\{i,j\}, (\tilde{y}, x_j(1)), \tilde{y} + x_j(0)) = (\tilde{y}, x_j(0))$. But since $x_j(1) > x_j(0) > 0$ and $\tilde{y} \geq x_i > 0$, this would violate LCSM-Resource. This proves the claim.

To determine ℓ, there are two sub-cases to consider: If $(j, x_j(1), x_j(0)) \sim (i, \hat{y}^{M-1}, x_i)$, then set $\ell = x_j(0)$. If $(j, x_j(1), x_j(0)) \succ (i, \hat{y}^{M-1}, x_i)$, then by Lemma 7 there exists a unique $\ell \in (x_j(0), x_j(1))$ such that $(j, x_j(1), \ell) \sim (i, \hat{y}^{M-1}, x_i)$. (Note that $\ell \neq x_j(1)$ since $\hat{y}^{M-1} \neq x_i$.)

For every $m \in \{0, 1, 2, \ldots, M\}$, set $y^m = \hat{y}^{M-m}$. Then $\{y^m\}_{m=0}^M$ and ℓ satisfy the desired requirements. □

For any $(i, c_i, x_i) \in Y^\circ$ and $j \in \{1,2,3\} \setminus \{i\}$, let $M_j(i, c_i, x_i)$ denote the M and $\ell_j(i, c_i, x_i)$ denote the ℓ from Lemma 21. The next lemma shows that the choice of j is without loss of generality.

Lemma 22. *For any $(i, c_i, x_i) \in Y^\circ$ and $j, k \in \{1,2,3\} \setminus \{i\}$, we have $M_j(i, c_i, x_i) = M_k(i, c_i, x_i)$ and $u_j(\ell_j(i, c_i, x_i)) = u_k(\ell_k(i, c_i, x_i))$.*

Proof. Abusing notation, let $M_j = M_j(i, c_i, x_i)$, $M_k = M_k(i, c_i, x_i)$, $\ell_j = \ell_j(i, c_i, x_i)$, and $\ell_k = \ell_k(i, c_i, x_i)$. Let $\{y_j^m\}_{m=0}^{M_j}$ and $\{y_k^m\}_{m=0}^{M_k}$ be the respective partitions of $[x_i, c_i]$ from Lemma 21. Since $(j, x_j(1), x_j(0)) \sim (k, x_k(1), x_k(0))$, Lemma 11 implies that we must have $M_j = M_k$ and $y_j^m = y_k^m$ for all m.

Simplifying notation, let $\{y^m\}_{m=0}^{M}$ denote the partition of $[x_i, c_i]$ now. Since $(i, y^1, y^0) \sim (j, x_j(1), \ell_j)$ and $(i, y^1, y^0) \sim (k, x_k(1), \ell_k)$, Lemma 21 implies $(j, x_j(1), \ell_j) \sim (k, x_k(1), \ell_k)$. Lemma 19 then implies $u_j(x_j(1)) - u_j(\ell_j) = u_k(x_k(1)) - u_k(\ell_k)$. But $u_j(x_j(1)) = u_k(x_k(1)) = 1$. Hence $u_j(\ell_j) = u_k(\ell_k)$. □

Our measure of a situation (i, c_i, x_i) is given by $M_j(i, c_i, x_i) - u_j(\ell_j(i, c_i, x_i))$, where $j \in \{1,2,3\} \setminus \{i\}$. The previous lemma shows this measure is independent of j. The final lemma of this subsection shows that this measure is additive.

Lemma 23. *Suppose $i \in \mathbb{N}$ and $c > b > a > 0$. Then for any $j \in \{1,2,3\} \setminus \{i\}$, we have*

$$M_j(i, c, b) - u_j(\ell_j(i, c, b)) + M_j(i, b, a) - u_j(\ell_j(i, b, a)) = M_j(i, c, a) - u_j(\ell_j(i, c, a)).$$

Proof. To simplify the proof, we will assume $1 = M_j(i, c, b) = M_j(i, b, a)$. Generalizing the proof is a straightforward but tedious exercise.

Set $\ell' = \ell_j(i, c, b)$ and $\ell'' = \ell_j(i, b, a)$. Thus we have $(i, c, b) \sim (j, x_j(1), \ell')$ and $(i, b, a) \sim (j, x_j(1), \ell'')$.

Case 1: $(j, x_j(1), x_j(0)) \succsim (i, c, a)$. If $(j, x_j(1), x_j(0)) \sim (i, c, a)$, then set $\hat{\ell} = x_j(0)$. Otherwise, if $(j, x_j(1), x_j(0)) \succ (i, c, a)$, then by Lemma 7 there exists a unique $\hat{\ell} \in (x_j(0), x_j(1))$ satisfying $(j, x_j(1), \hat{\ell}) \sim (i, c, a)$. Thus $M_j(i, c, a) = 1$ and $\ell_j(i, c, a) = \hat{\ell}$. Also, since $(i, c, b) \sim (j, x_j(1), \ell')$, Composition Down implies $(i, b, a) \sim (j, \ell', \hat{\ell})$. Since $(i, b, a) \sim (j, x_j(1), \ell'')$ by assumption, we have

$$(j, x_j(1), \ell'') \sim (i, b, a) \sim (j, \ell', \hat{\ell}).$$

Lemma 20 then implies $u_j(x_j(1)) - u_j(\ell'') = u_j(\ell') - u_j(\hat{\ell})$, or

$$u_j(\hat{\ell}) = u_j(\ell'') + u_j(\ell') - 1.$$

Thus we have:

$$\begin{aligned} M_j(i, c, a) - u_j(\ell_j(i, c, a)) &= 1 - u_j(\hat{\ell}) \\ &= 2 - u_j(\ell') - u_j(\ell'') \\ &= [1 - u_j(\ell')] + [1 - u_j(\ell'')] \\ &= [M_j(i, c, b) - u_j(\ell_j(i, c, b))] + [M_j(i, b, a) - u_j(\ell_j(i, b, a))]. \end{aligned}$$

Case 2: $(i,c,a) \succ (j,x_j(1),x_j(0))$. By Lemma 7, there exists a unique $y \in (a,c)$ such that $(i,c,y) \sim (j,x_j(1),x_j(0))$. (Note that $y < c$ since $x_j(1) > x_j(0)$.) In fact, it must be that $y \le b$ since $M_j(i,c,b) = 1$.

Case 2(a): $y = b$. Then we must have $\ell' = x_j(0)$ which implies $u_j(\ell') = 0$. Thus

$$M_j(i,c,b) - u_j(\ell_j(i,c,b)) + M_j(i,b,a) - u_j(\ell_j(i,b,a)) = 2 - u_j(\ell'').$$

Also, $\ell' = x_j(0)$ implies $(i,c,b) \sim (j,x_j(1),x_j(0))$. But since $(i,b,a) \sim (j,x_j(1),\ell'')$, this implies $M_j(i,c,a) = 2$ and $\ell_j(i,c,a) = \ell''$. Thus

$$M_j(i,c,a) - u_j(\ell_j(i,c,a)) = 2 - u(\ell'')$$

as desired.

Case 2(b): $y < b$. Then we must have $\ell' > x_j(0)$. Since $(i,c,y) \sim (j,x_j(1),x_j(0))$ and $(i,c,b) \sim (j,x_j(1),\ell')$, Composition Down implies $(i,b,y) \sim (j,\ell',x_j(0))$. Since $(j,x_j(1),\ell'') \sim (i,b,a)$ and $y > a$, item (i) of Lemma 9 implies there exists a unique $\hat{\ell} \in (\ell'',x_j(1))$ such that $(j,x_j(1),\hat{\ell}) \sim (i,b,y)$. Thus we have

$$(j,x_j(1),\hat{\ell}) \sim (i,b,y) \sim (j,\ell',x_j(0)).$$

Lemma 20 then implies $u_j(x_j(1)) - u_j(\hat{\ell}) = u_j(\ell') - u_j(x_j(0))$, or

$$1 - u_j(\hat{\ell}) = u_j(\ell'). \tag{4}$$

Furthermore, since $(j,x_j(1),\ell'') \sim (i,b,a)$ and $(j,x_j(1),\hat{\ell}) \sim (i,b,y)$, Composition Down implies $(j,\hat{\ell},\ell'') \sim (i,y,a)$. Since $(j,x_j(1),\ell'') \sim (i,b,a)$ and $y < b$, item (ii) of Lemma 9 implies there exists a unique $\hat{\ell}' \in (\ell'',x_j(1))$ such that $(j,x_j(1),\hat{\ell}') \sim (i,y,a)$. Thus we have

$$(j,x_j(1),\hat{\ell}') \sim (i,y,a) \sim (j,\hat{\ell},\ell'').$$

Lemma 20 then implies $u_j(x_j(1)) - u_j(\hat{\ell}') = u_j(\hat{\ell}) - u_j(\ell'')$, or

$$1 - u_j(\hat{\ell}') = u_j(\hat{\ell}) - u_j(\ell''). \tag{5}$$

To summarize, we have $(i,c,y) \sim (j,x_j(1),x_j(0))$, $(i,y,a) \sim (j,x_j(1),\hat{\ell}')$. Thus $M_j(i,c,a) = 2$ and $\ell_j(i,c,a) = \hat{\ell}'$. Using this, (4), and (5), we get:

$$\begin{aligned} M_j(i,c,a) - u_j(\ell_j(i,c,a)) &= 2 - u_j(\hat{\ell}') \\ &= 2 - u_j(\ell') - u_j(\ell'') \\ &= [1 - u_j(\ell')] + [1 - u_j(\ell'')] \\ &= [M_j(i,c,b) - u_j(\ell(i,c,b))] + [M_j(i,b,a) - u_j(\ell(i,b,a))]. \end{aligned}$$

□

B.3 Defining Utilities and Finishing the Proof

For any $i \in \mathbb{N}$ and $x > 0$, define

$$U_i(x) = \begin{cases} M_j(i,x,1) - u_j\left(\ell_j(i,x,1)\right) & \text{if } x > 1 \\ 0 & \text{if } x = 1 \\ u_j\left(\ell_j(i,1,x)\right) - M_j(i,1,x) & \text{if } x < 1, \end{cases}$$

where $j \in \{1,2,3\} \setminus \{i\}$. Lemma 22 implies that U_i is independent of j. Lemma 23 implies the following lemma.

Lemma 24. *For any $i \in \mathbb{N}$, $x > x' > 0$, and $j \in \{1,2,3\} \setminus \{i\}$, we have $U_i(x) - U_i(x') = M_j(i,x,x') - u_j(\ell_j(i,x,x'))$.*

The final four lemmas establish that $\{U_i\}_{i\in\mathbb{N}}$ satisfies all the conditions to be an equal sacrifice representation of S.

Lemma 25. *Suppose $(i,c_i,x_i),(j,c_j,x_j) \in Y$ and $i \neq j$. If $(i,c_i,x_i) \sim (j,c_j,x_j)$, then $U_i(c_i) - U_i(x_i) = U_j(c_j) - U_j(x_j)$.*

Proof. LCSM-Resource implies $c_i = x_i$ if and only if $c_j = x_j$. But in that case we would have $U_i(c_i) - U_i(x_i) = U_j(c_j) - U_j(x_j) = 0$.

Now suppose $c_i > x_i$ and $c_j > x_j$. Choose $k \in \{1,2,3\} \setminus \{i,j\}$. Set $\hat{M} = M_k(i,c_i,x_i)$ and $\hat{\ell} = \ell_k(i,c_i,x_i)$. Let $\{y^m\}_{m=0}^{\hat{M}}$ be the $\hat{M}$-partition of $[x_i,c_i]$ such that $(i,y^m,y^{m-1}) \sim (k,x_k(1),x_k(0))$ for every $m \in \{2,\dots,\hat{M}\}$, and $(i,y^1,y^0) \sim (k,x_k(1),\hat{\ell})$. Define $z^0 = x_j$ and $z^{\hat{M}} = c_j$. For $m \in \{1,2,\dots \hat{M}-1\}$, define z^m to be the unique award for j satisfying $(j,c_j,z^m) \sim (i,c_i,y^m)$. (Item (i) of Lemma 9 shows $\{z^m\}_{m=0}^{\hat{M}}$ is strictly increasing and unique since $(j,c_j,x_j) \sim (i,c_i,x_i)$ and $\{y^m\}_{m=0}^{\hat{M}}$ is strictly increasing.) Composition Down implies $(j,z^m,z^{m-1}) \sim (i,y^m,y^{m-1})$ for every $m \in \{1,2,\dots,M\}$. Lemma 11 then implies $(j,z^m,z^{m-1}) \sim (k,x_k(1),x_k(0))$ for every $m \in \{2,3,\dots,M\}$, and $(j,z^1,z^0) \sim (k,x_k(1),\hat{\ell})$. Hence $M(j,c_j,x_j) = \hat{M}$ and $\ell(j,c_j,x_j) = \hat{\ell}$. Lemma 24 then implies $U_i(c_i) - U_i(x_i) = \hat{M} - u_k(\hat{\ell}) = U_j(c_j) - U_j(x_j)$. ☐

Lemma 26. *Suppose $x = S(N,c,E)$. Then for every $i,j \in N$ such that $x_i, x_j > 0$, we have $U_i(c_i) - U_i(x_i) = U_j(c_j) - U_j(x_j)$.*

Proof. This is follows directly from Lemma 6 and Lemma 25. ☐

Lemma 27. *For every $i \in \mathbb{N}$, the function U_i is strictly increasing.*

Proof. By Lemma 24, U_i is strictly increasing if $M_j(i,x,x') - u_j(\ell_j(i,x,x')) > 0$ when $x > x' > 0$ and $j \in \{1,2,3\} \setminus \{i\}$. Note that $\ell_j(i,x,x') < x_j(1)$, which implies $u_j(\ell_j(i,x,x')) < 1$. Also $M_j(i,x,x') \geq 1$. Hence we must have $M_j(i,x,x') - u_j(\ell_j(i,x,x')) > 0$. ☐

Lemma 28. *For every $i \in \mathbb{N}$, the function U_i is continuous.*

Proof. This follows easily from Continuity and Lemma 26. ☐

Thus U_i is continuous and strictly increasing, and $\{U_i\}_{i\in\mathbb{N}}$ is an equal sacrifice representation of S.

C Proofs for Theorem 2 and Theorem 3

Proof of Theorem 2. Fix $b > a > 0$. Since U_i is strictly increasing for every i, we have $U_i(a) < U_i(b)$ for every i. Without loss of generality, assume $U_1(b) - U_1(a) \leq U_2(b) - U_2(a)$. (The proof is easily adapted if $U_1(b) - U_1(a) \geq U_2(b) - U_2(a)$.)

Define

$$\hat{U}_i(x) \equiv \frac{U_i(x) - U_i(a)}{U_1(b) - U_1(a)}.$$

and

$$\hat{V}_i(x) \equiv \frac{V_i(x) - V_i(a)}{V_1(b) - V_1(a)}$$

It is easy to verify that both $\hat{U}$ and $\hat{V}$ are equal sacrifice representations of S. Thus $\hat{V}_i$ and $\hat{U}_i$ are continuous and strictly increasing for every i. Also note that $\hat{U}_i(a) = 0$ for every i, $\hat{V}_i(a) = 0$ for every i, and $\hat{U}_1(b) = \hat{V}_1(b) = 1$. Also, the above assumption that $U_1(b) - U_1(a) \leq U_2(b) - U_2(a)$ implies $\hat{U}_1(b) = 1 \leq \hat{U}_2(b)$.

We will show that $\hat{U} = \hat{V}$. Once that is established, the proof is completed by setting

$$\alpha = \frac{V_1(b) - V_1(a)}{U_1(b) - U_1(a)}$$

and for every $i \in \mathbb{N}$

$$\beta_i = V_i(a) - \alpha U_i(a).$$

Step 1. $\hat{U}_1 = \hat{V}_1$.

By the Intermediate Value Theorem and strict monotonicity of $\hat{U}$, there exists $c_1 \in (a, b)$ such that $\hat{U}_1(c_1) = \frac{1}{2}$. Similarly since $\hat{U}_2(b) \geq 1$, there exists $c_2 \in (a, b)$ such that $\hat{U}_2(c_2) = \frac{1}{2}$. Since $\hat{U}$ is an equal sacrifice representation of S, we must have $S(\{1, 2\}, (c_1, c_2), 2a) = (a, a)$ and $S(\{1, 2\}, (b, c_2), c_1 + a) = (c_1, a)$. But since $\hat{V}$ is also an equal sacrifice representation of S, this means that

$$\hat{V}_1(c_1) - \hat{V}_1(a) = \hat{V}_2(c_2) - \hat{V}_2(a) = \hat{V}_1(b) - \hat{V}_1(c_1)$$

But since $\hat{V}_1(a) = 0$ and $\hat{V}_1(b) = 1$, this implies $\hat{V}_1(c_1) = \frac{1}{2} = \hat{U}_1(c_1)$.

By way of induction, fix $n \in \mathbb{N}$ and suppose for every $m \in [0, 2^n] \cap \mathbb{N}$ there exists $c_m \in [a, b]$ such that $\hat{U}_1(c_m) = \hat{V}_1(c_m) = \frac{m}{2^n}$. Now let $m \in [0, 2^{n+1}] \cap \mathbb{N}$. If m is even or zero, then $m = 2m'$ for some $m' \in [0, 2^n] \cap \mathbb{N}$. Hence by the inductive step there exists $c_{m'} \in [a, b]$ such that $\hat{U}_1(c_{m'}) = \hat{V}_1(c_{m'}) = \frac{m'}{2^n} = \frac{m}{2^{n+1}}$. If m is odd and non-zero, then $m - 1$ and $m + 1$ are even. Hence by the inductive step, there exists $c_{m-1}, c_{m+1} \in [a, b]$ such that $\hat{U}_1(c_{m-1}) = \hat{V}_1(c_{m-1}) = \frac{m-1}{2^{n+1}}$ and $\hat{U}_1(c_{m+1}) = \hat{V}_1(c_{m+1}) = \frac{m+1}{2^{n+1}}$. By the Intermediate Value Theorem and strict monotonicity of $\hat{U}$, there exists $c_m \in (c_{m-1}, c_{m+1}) \subset [1, 2]$ such that $\hat{U}_1(c_m) = \frac{m}{2^{n+1}}$. Similar to above, since $\hat{U}$ and $\hat{V}$ are both equal sacrifice representations of S, we must have $\hat{U}_1(c_m) = \hat{V}_1(c_m)$.

Since $\{\frac{m}{2^n} : n \in \mathbb{N}, m \in [0, 2^n] \cap \mathbb{N}\}$ is dense in $[0, 1]$ and since $\hat{U}_1$ and $\hat{V}_1$ are continuous and strictly increasing, we must have $\hat{U}_1 = \hat{V}_1$ on $[a, b]$.

Extending the result to all of $\mathbb{R}_{++}$ is straightforward since a and b were arbitrarily chosen. Finally we must have $\hat{U}_1(0) = \hat{V}_1(0)$ since they agree on $\mathbb{R}_{++}$ and are continuous.

Step 2. $\hat{U}_i = \hat{V}_i$ for all i.

By way of contradiction, suppose not. I.e. suppose there exists $i \neq 1$ and x such that $\hat{U}_i(x) \neq \hat{V}_i(x)$. Since $\hat{U}$ and $\hat{V}$ are both continuous, there exists y and $\epsilon > 0$ such that $\hat{U}_i(y) = \hat{V}_i(y)$ and $\hat{U}_i(z) \neq \hat{V}_i(z)$ for every $z \in (y, y+\epsilon)$. Since $\hat{U}_i$ is continuous and strictly increasing, there exists $z \in (y, y+\epsilon)$ such that $0 < \hat{U}_i(z) - \hat{U}_i(y) \leq 1$. The Intermediate Value Theorem implies there exists $c_1 \in [a, b]$ such that $\hat{U}_1(c_1) = \hat{U}_i(z) - \hat{U}_i(y)$. Since $\hat{U}$ is an equal sacrifice representation of S, we must have $S(\{1, i\}, (c_1, z), a+y) = (a, y)$. However note that $\hat{V}_i(z) - \hat{V}_i(y) \neq \hat{U}_i(z) - \hat{V}_i(y)$. Since $\hat{V}$ is also an equal sacrifice representation of S, then $S(\{1, i\}, (c_1, z), a+y) \neq (a, y)$ must hold, which is a contradiction. □

Proof of Theorem 3. Showing that the axioms are necessary is a straightforward exercise. So suppose S satisfies the stated axioms. By Theorem 1, we have $S \in \mathcal{ES}$. Let $U \in \mathcal{U}$ be the equal sacrifice representation of S. To show $S \in \mathcal{ES}^*$, Theorem 2 implies that it is sufficient to show that $U_i - U_j$ is constant for all i and j. But if i and j were such that $U_i - U_j$ was not constant, then one could easily construct a problem that violated Symmetry. □

D Proof of Theorem 4

First we state a standard result for concave functions.

Lemma 29. *A function $f : A \to \mathbb{R}$ is concave if and only if $f(a+h) - f(a) \geq f(b+h) - f(b)$ for $a < b$ and $h > 0$.*

The following lemma will be used in the proof.

Lemma 30. *Let $f : A \to \mathbb{R}$ be continuous. Suppose there exists $a < b$ and $\alpha \in (0,1)$ such that $(1-\alpha)f(a) + \alpha f(b) > f((1-\alpha)a + \alpha b)$. Then there exists $x \in (a, b)$ such that for every ϵ satisfying $0 < \epsilon < \min\{x - a, b - x\}$, we have $f(x) - f(x-\epsilon) < f(x+\epsilon) - f(x)$.*

Proof. Define the function $g : [0,1] \to \mathbb{R}$ to be

$$g(\beta) \equiv f((1-\beta)a + \beta b) - [(1-\beta)f(a) + \beta f(b)].$$

Note that $g(0) = g(1) = 0$, $g(\alpha) < 0$, and g is continuous. By the Extreme Value Theorem, g attains a global minimum on $[0,1]$. Define

$$\gamma \equiv \min\{\beta \in [0,1] : g(\beta) \leq g(\beta') \text{ for all } \beta' \in [0,1]\}.$$

Note that $\gamma \in (0,1)$ since $g(\alpha) < 0 = g(0) = g(1)$ and $\alpha \in (0,1)$.

Set

$$x \equiv (1-\gamma)a + \gamma b.$$

Note that $x \in (a, b)$. Now choose ϵ satisfying $0 < \epsilon < \min\{x - a, b - x\}$. Set $\beta' \equiv \frac{x-\epsilon-a}{b-a} = \gamma - \frac{\epsilon}{b-a}$ and $\beta'' \equiv \frac{x+\epsilon-a}{b-a} = \gamma + \frac{\epsilon}{b-a}$. Note then that $0 < \beta' < \gamma < \beta'' < 1$. Since γ is a global minimum of g, we have $g(\beta') > g(\gamma)$ and $g(\beta'') \geq g(\gamma)$. The first inequality implies

$$\begin{aligned} f(x-\epsilon) - [(1-\beta')f(a) + \beta' f(b)] &> f(x) - [(1-\gamma)f(a) + \gamma f(b)] \\ (\gamma - \beta')\,[f(b) - f(a)] &> f(x) - f(x-\epsilon), \end{aligned}$$

while the second inequality implies

$$\begin{aligned} f(x+\epsilon) - [(1-\beta'')f(a) + \beta'' f(b)] &\geq f(x) - [(1-\gamma)f(a) + \gamma f(b)] \\ f(x+\epsilon) - f(x) &\geq (\beta'' - \gamma)\,[f(b) - f(a)]\,. \end{aligned}$$

But since $\gamma - \beta' = \frac{\epsilon}{b-a} = \beta'' - \gamma$, this implies

$$f(x+\epsilon) - f(x) \geq \frac{\epsilon}{b-a}\,[f(b) - f(a)] > f(x) - f(x-\epsilon),$$

as desired. □

Now we turn to the proof of Theorem 4.

Proof of Theorem 4. ($\Leftarrow$) By assumption, S is an equal sacrifice rule with representation U, where U_i is concave for every i.

Fix the problem (N, c, E), $i \in N$, and $h > 0$. Set $x \equiv S(N, c, E)$ and $x' \equiv S(N, (c_i + h, c_{-i}), E + h)$. By way of contradiction, suppose $h < x'_i - x_i$. Since $x_i + h < x'_i$, this implies that there exists $j \in N \setminus \{i\}$ such that $x'_j < x_j$. Consistency implies $(x_i, x_j) = S(\{i, j\}, (c_i, c_j), x_i + x_j)$ and $(x'_i, x'_j) = S(\{i, j\}, (c_i + h, c_j), x'_i + x'_j)$. Since $x'_i > 0$, $x_j > 0$, and U is an equal sacrifice representation of S, we must have

$$U_i(c_i + h) - U_i(x'_i) \geq U_j(c_j) - U_j(x'_j)$$

and

$$U_j(c_j) - U_j(x_j) \geq U_i(c_i) - U_i(x_i).$$

Also, since U_i is concave and $h > 0$, Lemma 29 implies

$$U_i(c_i) - U_i(x_i) \geq U_i(c_i + h) - U_i(x_i + h).$$

Finally, since U_i is strictly increasing, we have

$$U_i(c_i + h) - U_i(x_i + h) > U_i(c_i + h) - U_i(x'_i).$$

Putting this all together, we have

$$U_j(c_j) - U_j(x_j) > U_j(c_j) - U_j(x'_j).$$

But this implies $U_j(x_j) < U_j(x'_j)$, or $x_j < x'_j$ since U_j is strictly increasing. This contradicts $x'_j < x_j$.

($\Rightarrow$) By assumption, S satisfies Linked Claims-Resource Monotonicity and is an equal sacrifice rule with representation U. By way of contradiction, suppose there exists i such that U_i is not concave. I.e. there exists $b > a > 0$ and $\alpha \in (0, 1)$ such that $(1 - \alpha)U_i(a) + \alpha U_i(b) > U_i((1 - \alpha)a + \alpha b)$. By Lemma 30, there exists $c_i \in (a, b)$ such that for every δ satisfying $0 < \delta < \min\{c_i - a, b - c_i\}$, we have $U_i(c_i) - U_i(c_i - \delta) < U_i(c_i + \delta) - U_i(c_i)$. Choose $h' < \min\{c_i - a, b - c_i\}$.

Fix $j \neq i$ and $c_j > 0$. Choose $\epsilon < \min\{U_i(c_i) - U_i(c_i - h'), \lim_{a \to 0} U_j(c_j) - U_j(a)\}$. Since U_i and U_j are continuous and strictly increasing, there exists $x_i \in (0, c_i)$ and $x_j \in (0, c_j)$ such that $U_j(c_j) - U_j(x_j) = U_i(c_i) - U_i(x_i) = \epsilon$. Note that this implies $S(\{i, j\}, (c_i, c_j), x_i + x_j) = (x_i, x_j)$. Set $h \equiv c_i - x_i$. Thus we have $x_i = c_i - h$ and $x_i, c_i + h \in (a, b)$. By Lemma 30, $U_i(c_i) - U_i(x_i) < U_i(c_i + h) - U_i(c_i)$. But then this implies $U_i(c_i + h) - U_i(c_i) > U_j(c_j) - U_j(x_j)$. Since U_i and U_j are both continuous and strictly increasing, this means that $S_i(\{i, j\}, (c_i + h, c_j), c_i + x_j) > c_i = x_i + h$. But this implies $S_i(\{i, j\}, (c_i + h, c_j), c_i + x_j) - S_i(\{i, j\}, (c_i, c_j), x_i + x_j) > h$, which violates Linked Claims-Resource Monotonicity. □

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