To Dissimulate or Not to Dissimulate? Insider Trading When Anticipating Future Information

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Abstract

We analyze a dynamic model of a monopolistic insider who receives private information sequentially and faces a post-trading disclosure requirement. We show that characterizing the equilibrium in this trading game is isomorphic to solving a consumption-saving problem with a borrowing constraint. Analogous to the "consumption-smoothing" intuition in the consumption-saving literature, the insider in our trading game "smooths" his information usage over time given the dynamics of his private information. The insider would "dissimulate" his private information through mixed strategies if and only if sufficient information arrives early. Finally, we analyze the interpretation of mixed strategies and the value of commitment.

Keywords: Insider trading; Dissimulation; Mixed strategy; Information arrival.

JEL classification: C72, D82, G14

1. Introduction

How insiders trade in financial markets and affect prices is a central question in economics. Motivated by the post-trade disclosure requirement on insiders, Huddart et al. (2001) (HHL (2001) henceforth) extend the model in Kyle (1985) so that the insider needs to disclose his trade after each transaction. Disclosure changes the nature of information revelation in the market since the insider's trading can no longer "hide" behind noise trading. The key insight in HHL (2001) is that the insider "dissimulates" his private information by adopting a mixed strategy in early rounds so he can exploit the information in later rounds. In equilibrium, the insider utilizes his information at a constant rate in the sense that the same amount of private information is revealed each period.

The literature so far has abstracted away from the fact that an insider typically obtains information on an ongoing basis. How does this feature affect the insider's trading strategy and asset prices? For example, how does the anticipation of future information affect the insider's trading strategy today? As noted in the literature, an insider can dissimulate his current long-lived information to use in the future. That is, one can "transfer" his current information to future periods. However, one cannot transfer his future information to utilize today. Thus, the technology for transferring information across time is "asymmetrical." How does this asymmetry affect the insider's trading strategy and asset prices? Our paper tries to answer these questions.

Our model is an extension of Kyle (1985) and HHL (2001). Specifically, we consider an N-period model with one risky asset and one monopolistic insider. The risky asset is a claim to an uncertain cash flow in period N. Each period the insider is endowed with some long-lived private information, which is a private signal about the risky asset's liquidation value in period N. The insider trades against noise traders and a risk-neutral market maker sets the price. As in HHL (2001), the insider faces a post-trade disclosure requirement, that is, he needs to disclose his trade after his transaction each period.[1]

In each period, the insider decides on how aggressively he should exploit his current information while taking into account two features. First, as in HHL (2001), the post-trade disclosure would completely reveal his current information unless he dissimulates his information by adopting a mixed strategy. Second, he expects to receive further private information in future periods. The anticipation of the amount of future private information naturally affects the insider's trading today.

Our paper has three main contributions. First, we provide a methodology to compute the linear equilibrium in the dynamic trading game by showing that solving the linear equilibrium—which is a general equilibrium concept that involves the interactions between the insider and the market maker—is isomorphic to computing a standard consumption-saving problem with a borrowing constraint—which is a simple partial equilibrium concept.

Specifically, we reduce the equilibrium characterization to solving the insider's optimal information usage problem. We show that this information usage problem can be mapped to a consumption-saving problem of an agent with a power utility. Each period, the insider's private signal and his information usage in our original model corresponds to the agent's income and consumption, respectively. The aforementioned asymmetry in information transfer technology corresponds to the agent's technology in transferring consumption over time: the agent can transfer his current consumption to future periods but the not the other away around. That is, the interest rate is zero and the agent can save but not borrow.

Equipped with this mapping, we can figure out the linear equilibrium in our trading game using the standard dynamic programming approach, with a state variable being the amount of unused information by the insider till period n. In particular, the value of this state variable governs whether the insider dissimulates or not in equilibrium: he plays a mixed strategy in period n if and only if the state variable is positive.

[1] See, for example, Section 16(a) and 13(d) requirements of Securities Exchange Act of 1934 and amendment by Sarbanes-Oxley Act of 2002.

Our second contribution is that, inspired by the "consumption-smoothing" intuition in a consumption-saving problem, we show that the investor's optimal trading strategy minimizes the variation of his information usage over time. That is, the traditional consumption-smoothing result corresponds to the "information-usage-smoothing" result in our model. We also show that smoothing information usage allows the insider to smooth his price impact over time. Ideally, the insider would like to "walk down the demand curve" and have the same price impact each period, which is equivalent to utilizing the same amount of information each period. However, this is not always possible because of the asymmetry reflected in the insider's information budget constraint. As noted earlier, the insider can transfer his current information to future periods but not the other way around. Hence, if most of the private information arrives in early periods, the insider can perfectly smooth his information over time, i.e., utilize the same amount of information each period, implying that the insider does not fully use up his private information by playing a mixed strategy in early periods. However, if most information arrives in late periods, it is infeasible for the insider to have a constant information usage rate each period because he cannot transfer information in late periods to early ones. Instead, the insider has to use up his information endowment and play a pure strategy in early periods.

These results extends and sharpens the insight in HHL (2001), who focus on the special case in which the insider possesses all his private information in the first period and hence achieves perfect information smoothing through mixed strategies. In contrast, if the insider expects to receive a large amount of private information in the future, he would exploit his current information more aggressively. In fact, if future information is sufficiently abundant, he would adopt a pure strategy, which would fully reveal his current information after disclosure. This result is consistent with the evidence in Koudijs (2015), who studies insider trading in the 18^{th} century in Amsterdam where private information arrives from London via sailing boats and finds that an informed investor would trade aggressively if he expects the next boat to arrive shortly.

Finally, to interpret the insider's mixed strategy, we consider an alternative setup whereby the insider commits to a strategy of adding noise to his demand and consciously chooses the variance of the noise optimally. The committed strategy can be viewed as a predetermined trading plan implemented by an algorithm. The rest of the setup remains identical to our baseline model. We find that the equilibrium in this variation model with commitment is identical to that in our baseline model. That is, the implications on whether to dissimulate and how much noise to add to demand are the same across the two models. Hence, the mixed strategy in our model can be viewed as the insider actively choosing how much information to dissimulate each period.

Our paper adds to the literature on informed trading by corporate insiders and institutional investors in financial markets. This literature is voluminous and so we discuss most related studies, organized according to the two important features in our setting: the insider's trade must be disclosed after the fact and so he might play a mixed strategy; and the insider acquires information sequentially. In terms of the first feature, our paper is most related to HHL (2001), who is the first study to constructively demonstrate that, in a Kyle (1985) model, the insider can play a mixed strategy when his trade is mandated to be disclosed. Yang and Zhu (2020) investigate the behavior of an insider who leaks a signal about the demand to back-runners. The insider can choose between the pure and mixed strategies, and is more likely to choose the latter if the information leakage is more severe. Back and Baruch (2004) present in a variant of Glosten and Milgrom (1985) model that an insider would take mixed strategy by randomizing over trades to buy, sell, and wait, and they show its convergence to the Kyle-style equilibrium insider strategy. Our paper complements these studies in several ways. First, methodology wise, we transform the equilirbium characterization into a simple consumption-saving problem. Second, we extend and sharpen the results in HHL (2001) and characterize when the insider dissimulates in equilibrium. Third, we provide an interpretation of the mixed strategy played in a Kyle (1985) model and also show that commitment has no value in the linear equilibrium.

In terms of the second feature of sequential information arrival, we believe that this feature is relevant to many settings in practice. The information acquisition process may result from the dynamics of informational events—such as IPO (e.g., Welch, 1992; Lowry and Schwert, 2002), acquisition (e.g., Denis and Macias, 2013), and mergers (e.g. Ferreira and Laux, 2007)—or/and the dynamics of research and learning generating private information flow (e.g. Banerjee and Breon-Drish, 2022; Johannes et al., 2014). Bernhardt and Miao (2004), Caldentey and Stacchetti (2010) and Sastry and Thompson (2019) examine the impacts of sequential information acquisition. The public disclosure requirement and the associated mixed strategy equilibrium features distinguish our study from theirs.

2. Model

Our model setup is parallel to that in HHL (2001). The only difference is the assumption on the insider's information. In HHL (2001), the insider obtains all his private information about the asset's liquidation value at the initial period and receives no further private information afterwards. In contrast, our analysis focuses on the sequential arrival of private information.

The economy has one risky asset and lasts for N periods, denoted by $n = 1, ..., N$. The risky asset has a liquidation value at the final period N, which is denoted as F and has an ex ante distribution of $\mathcal{N}(0, \sigma_F^2)$ with $\sigma_F > 0$. The market is populated by an insider, a continuum of noise traders, and a market maker. Everyone is risk neutral. The insider submits a market order to trade x_n shares in period n. The market maker sets the asset price to break even. The time line of events in period n is summarized in Figure 1.

The insider observes private information about the asset's liquidation value and, critical to our analysis, his information arrives over time. To capture this sequential learning feature,

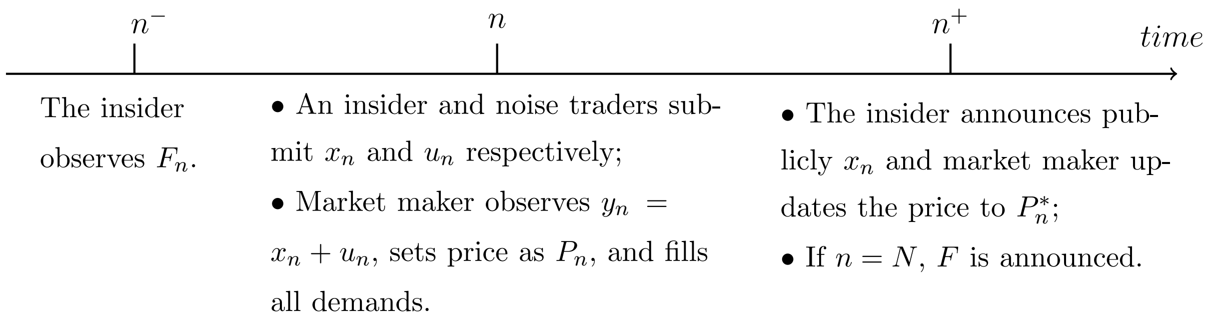


Figure 1. Timeline

we divide the asset's liquidation value F into N elements as follows:

$$F = \sum_{n=1}^{N} F_n$$

where $F_n \sim \mathcal{N}(0, \sigma_{F_n}^2)$ with $\sigma_{F_n} \geq 0$ and is serially independent across n. By construction, $\sigma_F^2 = \sum\limits_{n=1}^{N} \sigma_{F_n}^2$. Prior to each trading time $n = 1, ..., N$ (denoted by n^- in Figure 1), the insider observes F_n. Note that F_n is long-lived information in the sense that it affects the asset's final liquidation value and never becomes public before the final period. If $\sigma_{F_1}^2 = \sigma_F^2$ and $\sigma_{F_n}^2 = 0$ for $n > 1$, then our setting degenerates to HHL as the insider receives all his private information in the first period.

Noise traders have an aggregate demand of u_n shares, with $u_n \sim \mathcal{N}(0, \sigma_u^2)$ (with $\sigma_u > 0$) and u_n is independent across n and from F_n. As standard in the literature, noise trading provides the randomness to hide the insider's trade from the market maker. Upon receiving the aggregate order flow from the insider and noise traders, $y_n = x_n + u_n$, the market maker sets the price P_n to his expectation of the liquidation value to execute the trade (the trading time is denoted by n in Figure 1). As in HHL, the insider is required to disclose his trading ex post. That is, after the transaction at period n but before the next trading period $n + 1$ (denoted by n^+ in Figure 1), the insider publicly announces his trade size x_n, and based on this announcement, the market maker adjusts her break-even price from P_n to P_n^*.

Formally, in period n, the market maker's information set is $\mathcal{I}_n^M \equiv \{y_1, ..., y_n, x_1, ..., x_{n-1}\}$

at the time of trading and is $\mathcal{I}_{n+}^M \equiv \{y_1, ..., y_n, x_1, ..., x_n\}$ after the insider's disclosure of his trade x_n. At the time of transaction, the market maker sets the execution price to

$$P_n = E[F|\mathcal{I}_n^M]. \tag{1}$$

After the insider's disclosure, the market maker adjusts the asset price to

$$P_n^* = E[F|\mathcal{I}_{n+}^M]. \tag{2}$$

When computing prices P_n and P_n^* in equations (1) and (2), the market maker takes as given the insider's trading strategies.

The insider's information set in period n is $\mathcal{I}_n^I \equiv \{F_1, ..., F_n, P_1, ..., P_{n-1}, P_1^*, ..., P_{n-1}^*\}$. The insider maximize his expected trading profits:

$$\max_{x_n,\ldots,x_N} E\left[\sum_{j=n}^{N} \pi_j|\mathcal{I}_n^I\right], \tag{3}$$

where $\pi_j \equiv x_j(F - P_j)$ is his trading profit directly attributable to his period j trade. In computing his optimal trade in (3), the insider takes the market maker's pricing rules as given. Following Kyle (1985), we define an equilibrium as follows:

Definition 1. An equilibrium is defined as trading strategies and pricing rules (x_n, P_n, P_n^*), for $n = 1, ..., N$, such that at period n: (a) the market maker sets prices according to (1) and (2), taking the insider's trading strategies as given; and (b) the insider's strategy $\{x_n, ...x_N\}$ solves (3), taking the market maker's pricing rules as given.

3. Analysis

In this section, we first characterize the equilibrium in our setting and show that computing the equilibrium in our trading game (which is a general equilibrium concept) is equivalent to solving a standard borrowing constrained consumption-saving problem (which is a partial equilibrium concept). We then present a recursive formulation of the insider's problem and characterize when the insider plays a pure or mixed strategy in equilibrium.

3.1 Equilibrium Characterization: An Equivalence Result

We follow Kyle (1985) and HHL (2001) and consider linear equilibria. That is, in period n, for $n = 1, ..., N$, the trading strategies and the pricing rules are given by

$$x_n = \beta_n(\sum_{i=1}^{n} F_i - P^*_{n-1}) + z_n, \tag{4}$$

$$P_n = P^*_{n-1} + \lambda_n y_n, \tag{5}$$

$$P^*_n = P^*_{n-1} + \gamma_n x_n, \tag{6}$$

where $z_n \sim \mathcal{N}(0, \sigma^2_{z_n})$, $P^*_0 = 0$, and the parameters $\{\beta_n,\ \lambda_n,\ \gamma_n,\ \sigma_{z_n}\}$ are determined in equilibrium.

Intuitively, the insider observes private information F_n in period n and hence $\sum_{i=1}^{n} F_i - P^*_{n-1}$ is the difference between the insider's expected liquidation value computed based on his private information and the asset price determined based on the public information. So, this difference captures the insider's information advantage relative to the market and his trade is linear in this difference in (4). Moreover, as pointed out in HHL (2001), due to the disclosure requirement, the insider may play a mixed strategy, i.e., add noise to his trade to dissimulate his private information. Mathematically, in period n, the insider adopts a mixed strategy if $\sigma_{z_n} > 0$ and a pure strategy if $\sigma_{z_n} = 0$. The pricing function in (5) reflects that the market maker adjusts the execution price in period n based on the aggregate order flow y_n. After the insider's disclosure, as shown in (6), the market maker further adjusts the asset price based on the insider's trade x_n.

The following theorem characterizes the unique linear equilibrium. As a methodological contribution, this theorem also demonstrates that solving a linear equilibrium in our economy is equivalent to solving a standard borrowing-constrained consumption-saving problem.

Theorem 1. *There exists a unique linear equilibrium in which the insider's trading strategies and the market maker's pricing rules are given by equations (4)–(6) with parameters* *characterized as follows: For* $n = 1, ..., N$,

$$\beta_n = \frac{k_n \sigma_u}{\Sigma_n + k_n^2}, \tag{7}$$

$$\lambda_n = \frac{k_n}{2\sigma_u}, \tag{8}$$

$$\gamma_n = \frac{k_n}{\sigma_u}, \tag{9}$$

$$\sigma_{z_n}^2 = \frac{\Sigma_n}{\Sigma_n + k_n^2} \sigma_u^2, \tag{10}$$

where

$$\Sigma_n = \sum_{i=1}^{n} \sigma_{F_i}^2 - \sum_{i=1}^{n} k_i^2, \tag{11}$$

and $\{k_1, \cdots, k_N\} \in \mathbb{R}_{\geq 0}^N$ *are the unique non-negative solution to the following maximization problem:*

$$\max_{\{k_1, \cdots, k_N\} \in \mathbb{R}_{\geq 0}^N} (k_1 + ... + k_N), \tag{12}$$

$$\textit{subject to} \qquad \sum_{i=1}^{n} k_i^2 \leq \sum_{i=1}^{n} \sigma_{F_i}^2, \ \textit{for}\ n = 1, ..., N. \tag{13}$$

The above theorem shows that all equilibrium parameter values are pinned down by equations (7)–(11) once the value of k_n is determined. The values of $\{k_1, \cdots, k_N\}$ are computed further from the constrained maximization problem described in (12) and (13).

In this constrained maximization problem, the choice variable k_n^2 represents how much information the insider utilizes in each period. In the proof of Theorem 1, we show that variable k_n^2 is the variance of the asset price change in period n:

$$k_n^2 = Var(P_n^* - P_{n-1}^*). \tag{14}$$

Empirically, k_n corresponds to return volatility. Nonetheless, in our analysis, we emphasize more its interpretation from an information perspective. Recall that the risk-neutral market maker sets the price P_n^* as the expected asset liquidation value based on aggregate order flows and insider trading disclosure. So, the price process is a martingale and price changes are independent over time. The variance of these changes is driven by the new information revealed by the insider's period-n trade. If the insider trades more aggressively in period n and uses more of his private information, then $P_n^* - P_{n-1}^*$ reveals more new information and its variance k_n^2 is higher.

Conditions in (13) are the insider's "information budget constraint." In each period n, the total amount of information the insider has utilized during periods 1 through n should be no more than the total amount of information he has received by then.

The objective in (12) is essentially the sum of the insider's expected profits across N periods. Specifically, since the market maker is risk neutral and breaks even, the insider's expected profits in period n must be equal to the noise trader's expected loss in period n: $E[\pi_n] = \lambda_n \sigma_u^2$. Combined with the expression of λ_n in equation (8) in Theorem 1, we obtain $E[\pi_n] = \frac{k_n}{2}\sigma_u$. Hence, the insider's expected trading profit in period n is a product of his information usage k_n in the period and the volatility of noise trader's demand σ_u.

We can reinterpret the constrained maximization problem in (12) and (13) as a standard textbook consumption-saving problem subject to a borrowing constraint. Specifically, we can consider an agent who lives for N periods and consumes $C_n \geq 0$ in period n. This agent's flow utility in period n is $u(C_n) = \sqrt{C_n}$. He does not discount future so that his life time utility is $\sum_{n=1}^{N} u(C_n)$. The agent receives deterministic endowment income $w_n \geq 0$ in period n. He can save at a net interest rate of 0 but cannot borrow. Then this agent's consumption-saving problem is as follows:

$$\max_{\{C_1,\cdots,C_N\}\in\mathbb{R}^N_{\geq 0}} (\sqrt{C_1} + ... + \sqrt{C_N}),$$

$$\text{subject to} \qquad \sum_{i=1}^{n} C_i \leq \sum_{i=1}^{n} w_i, \text{ for } n = 1, ..., N.$$

We can see that the above problem is isomorphic to the constrained maximization problem in (12) and (13) by relabeling k_n^2 as C_n and σ_F^2 as w_n.

3.2 To Dissimulate or Not? A Recursive Formulation

Given the mapping between the equilibrium characterization of our trading game and the standard borrowing-constrained consumption-saving problem, we can follow the standard steps in textbooks such as Merton (1992) and formulate the insider's information usage problem defined in (12) and (13) into a dynamic programming problem. To achieve this representation, we introduce a state variable Σ_n, which measures the amount of private information not used by the insider till period n. This variable Σ_n would be the counterpart of the beginning period wealth in the dynamic programming representation for the standard consumption-saving problem.

Formally, variable Σ_n is defined as follows:

$$\Sigma_n \equiv Var\left(\sum_{i=1}^{n} F_i | P_1^*, ..., P_n^*\right).$$

Note that $\sum\limits_{i=1}^{n} F_i$ is the total private information possessed by the insider till period n. With trade and disclosure, the prices $\{P_n^*\}$ set by the market maker reveal some of this private information (see the insider's trading strategy in (4) and the market maker's pricing rule in (6)). If the prices reveal much information about the insider's private information, then the variance Σ_n is small. In particular, if the prices reveal all the insider's private information till period n, then this variance degenerates to 0. In fact, variable Σ_n determines whether the insider plays a pure or mixed strategy in equilibrium (i.e., whether $\sigma_{z_n} = 0$ or $\sigma_{z_n} > 0$). By equation (10), the insider plays a pure strategy in equilibrium if and only if if $\Sigma_n = 0$.

At the beginning of period n, the insider inherits from the previous period a stock Σ_{n-1} of remaining balance of unused private information. He then observes private signal F_n (which has a variance of $\sigma_{F_n}^2$) and so his balance of unused information increases by $\sigma_{F_n}^2$, resulting in a total balance of $\Sigma_{n-1} + \sigma^2_{F_n}$ prior to trading. When determining his information usage k_n^2 in the period-n trading, the insider faces the following constraint:

$$k_n^2 \leq \Sigma_{n-1} + \sigma^2_{F_n}. \tag{15}$$

The above inequality captures the asymmetrical information transfer technology mentioned in the introduction. That is, the insider is constrained to use the private information only up to period n, which, in relation to the consumption-saving analogy, corresponds to the specification that the agent can only save the current wealth to the future at a zero interest rate but cannot borrow income from the future.

If the insider plays a pure strategy in period n, then he uses up his information and chooses $k_n^2 = \Sigma_{n-1} + \sigma^2_{F_n}$, leaving a zero balance of unused private information after period n. If the insider plays a mixed strategy in period n, then the market maker cannot infer from the disclosed x_n all the private information possessed by the insider. In this case, the insider chooses $k_n^2 < \Sigma_{n-1} + \sigma^2_{F_n}$, leaving a positive balance of unused information to the next period. In both cases, the amount of unused private information Σ_n evolves according to

$$\Sigma_n = \Sigma_{n-1} + \sigma^2_{F_n} - k_n^2, \tag{16}$$

for $n = 1, ..., N$, with $\Sigma_0 \equiv 0$. With the aid of the budget constraint for the choice variable k_n^2 given by (15) and the dynamics of the state variable Σ_n given by (16), we can write down the dynamic programming representation of the insider's information-usage problem in (12) and (13), which is formalized in the following proposition.

Proposition 1. *The equilibrium information usage k_n is determined by the following dynamic programming problem:*

$$V_n(\Sigma_{n-1}) = \max_{k_n}[k_n + V_{n+1}(\Sigma_n)], \text{ for } n = 1, ..., N,$$

subject to (15) and (16), with terminal value $V_{N+1} = 0$. If $\Sigma_n = 0$, then in period n the *insider is playing a pure strategy in equilibrium. If $\Sigma_n > 0$, then in period n the insider is playing a mixed strategy in equilibrium.*

Example: The Two-Period Case. We conclude this section by considering the case of $N = 2$ as an example. That is, there are two rounds of trading and the insider receives his private signals F_1 and F_2 before the first and second trading periods, respectively. As mentioned before, our model nests the two-period model in HHL (2001) as a special case. In HHL (2001), the insider receives only one signal before the first round of trading. Hence, if we set $\sigma_{F_2} = 0$ in our model, our set-up is identical to the two-period model in HHL (2001). The following proposition characterizes the equilibrium in our model for the case of $N = 2$.

Proposition 2. *If $N = 2$, the equilibrium is characterized in the following two cases:*

Case 1: If $\sigma^2_{F_1} > \sigma^2_{F_2}$, then the insider plays a mixed strategy in period 1 and a pure strategy in period 2, and the equilibrium is given by

$$\sigma^2_{z_1} = \frac{\sigma^2_{F_1} - \sigma^2_{F_2}}{2\sigma^2_{F_1}}\sigma^2_u, \quad \sigma^2_{z_2} = 0,$$
$$\beta_1 = \frac{\sigma_F \sigma_u}{\sqrt{2}\sigma_{F_1^2}}, \quad \beta_2 = \frac{\sqrt{2}\sigma_u}{\sigma_F}, \quad k_1 = k_2 = \frac{\sigma_F}{\sqrt{2}},$$
$$\lambda_1 = \lambda_2 = \frac{\sigma_F}{2\sqrt{2}\sigma_u}, \quad \gamma_1 = \gamma_2 = \frac{\sigma_F}{\sqrt{2}\sigma_u}.$$

Case 2: If $\sigma^2_{F_1} \leq \sigma^2_{F_2}$, then the insider plays a pure strategy in both periods, and the equilibrium is given by

$$\sigma^2_{z_1} = \sigma^2_{z_2} = 0,$$
$$\beta_i = \frac{\sigma_u}{\sigma_{F_i}}, \quad \lambda_i = \frac{\sigma_{F_i}}{2\sigma_u}, \quad \gamma_i = \frac{\sigma_{F_i}}{\sigma_u}, \quad k_i = \sigma_{F_i}, \quad \textit{for } i = 1, 2.$$

We observe that whether the insider plays a mixed strategy in period 1 depends on whether he receives more private information in the first period than in the second (i.e., $\sigma_{z_1} > 0$ if and only if $\sigma_{F_1} > \sigma_{F_2}$). This result can be intuitively understood from our consumption-saving analogy. For instance, if the agent's total income across two periods is \$1, then the convexity of preferences, i.e., the concavity of the utility function $u(C_n)$, would push the agent to allocate his total wealth equally across two periods, i.e., $C_n^* = \$0.5$. However, whether this equal allocation is feasible depends on the income pattern since the agent can only save but not borrow.

Suppose that the agent obtains the total life time income of \$1 from receiving \$0.7 in period 1 and \$0.3 in period 2, then he can achieve equal allocation, that is, he will consume $C_1^* = \$0.5$ out of the first period income \$0.7, and save the remaining \$0.2 till next period, so that he can also consume $C_2^* = \$0.5$ in period 2 from the saving of \$0.2 and the second period income of \$0.3. If he receives \$0.3 in period 1 and \$0.7 in period 2, then he will be forced to consume his endowments, i.e., $C_1^* = \$0.3$ and $C_2^* = \$0.7$, because he cannot borrow to consume the ideal amount.

We can recast the above analogy intuition directly into our setting. Parallel to the instance above, now we endow the insider with a total amount of information of 1 (i.e., $\sigma_{F_1}^2 + \sigma_{F_2}^2 = 1$). The insider will use up all the information after trading in period 2, that is, $k_1^2 + k_2^2 = 1$. Recall that the insider's objective is to maximize $k_1 + k_2$. The convexity of the quadratic function k^2 would push the insider to allocate equal information usage across periods, that is, $k_1^{2*} = k_2^{2*} = 0.5$. Again, whether this ideal allocation is feasible depends on the information arrival pattern.

If the insider receives more information in the first period, say, $\sigma_{F_1}^2 = 0.7$ and $\sigma_{F_2}^2 = 0.3$ (which is Case 1 in Proposition 2), then he can achieve his ideal allocation. That is, he will choose $k_1^{2*} = 0.5$ in period 1 and leave a balance 0.2 of unused information amount to the next period (i.e., $\Sigma_1 = \sigma_{F_1}^2 - k_1^{2*} = 0.7 - 0.5 = 0.2$), so that in the next period he can choose $k_2^{2*} = 0.5$ (i.e., $k_2^{2*} = \Sigma_1 + \sigma_{F_2}^2 = 0.2 + 0.3 = 0.5$). This corresponds to that the insider plays a mixed strategy in period 1 since he has kept some of his period-1 private information for the second period. By contrast, if the insider receives less information in the first period, for instance, $\sigma_{F_1}^2 = 0.3$ and $\sigma_{F_2}^2 = 0.7$ (which is Case 2 in Proposition 2), then the insider will use up his private information in period 1 (i.e., $k_1^{2*} = 0.3$ and $k_2^{2*} = 0.7$). In this case, he is playing a pure strategy in both periods.

Our results in Proposition 2 generalize and sharpen those in HHL (2001). The model in HHL (2001) belongs to Case 1 with $\sigma_{F_1}^2 = \sigma_F^2$ and $\sigma_{F_2}^2 = 0$. Our analysis in Case 1 shows that the dissimulation result in HHL (2001) hold more generally, i.e., as long as $\sigma_{F_1}^2 > \sigma_{F_2}^2$. The dissimulation result, however, disappears in Case 2, where the insider receives less private information in the first period than in the second ($\sigma_{F_1}^2 < \sigma_{F_2}^2$). Anticipating the arrival of more information in the second period, the insider does not dissimulate his information in the first period. Instead, he utilizes all his private information available at that time.

4. Smoothing Information Usage

In this section, we first explore the insider's incentive to smooth information usage over time, and in particular, we ask when perfect smoothing, as analyzed in HHL (2001), is possible. We then study settings featuring monotonic information arrivals, which allows analytical characterization of the insider's information usage choices.

4.1 When Can the Insider Achieve Perfect Smoothing?

In the discussion of the two-period case, we observe that the insider prefers to smooth his information usage over time. We can formally show that the insider aims to minimize the time variation of his information usage across periods. Also, minimizing the variations in information usage over time is equivalent to minimizing the variations in price impact. Intuitively, the insider's private information usage is closely linked to his price impact, and indeed, these two are proportional to each other in our model. So, smoothing information usage across periods is the same as smoothing price impact over time. Formally, we have the following proposition.

Proposition 3. *The insider's information usage problem defined in (12)-(13) and the following two minimization problems are equivalent to each other:*

(1) Smoothing information leakage over time:

$$\min_{\{k_1,\cdots,k_N\}\in\mathbb{R}^N_{\geq 0}} (k_1-\overline{k})^2+...+(k_N-\overline{k})^2,$$

subject to the information budge constraints (13) with the final one being equality, where $\overline{k}\equiv(k_1+...+k_N)/N$.

(2) Smoothing price impact over time:

$$\min_{\{\lambda_1,\cdots,\lambda_N\}\in\mathbb{R}^N_{\geq 0}} (\lambda_1-\overline{\lambda})^2+...+(\lambda_N-\overline{\lambda})^2,$$

subject to the information budge constraints (13) with the final one being equality, where $\overline{\lambda}\equiv(\lambda_1+...+\lambda_N)/N$.

The equivalence result presented in Proposition 3 generalizes the insight in Kyle (1985) and HHL (2001) who show that to maximize his expected trading profit, a monopolistic insider minimizes the time variation of his information usage to zero, i.e, utilizes the same amount of information each period. Since the total amount of the insider's private information is σ_F^2, the best possible scenario is to utilize the same amount (i.e., σ_F^2/N) of information each period. This strategy, however, is not always feasible in our model, as we discussed in the two-period example. When is this perfect smoothing of information usage possible? The answer is provided in the following proposition.

Proposition 4. *The necessary and sufficient condition for perfect smoothing of information usage (i.e.,* $k_n^{2*}=\sigma_F^2/N$ *for* $n=1,\cdots,N$*) is*

$$\sum_{i=1}^{n}\sigma_{F_i}^2\geq\frac{n}{N}\sigma_F^2,\ \text{for } n=1,\cdots,N. \tag{17}$$

Under condition (17), sufficient private information arrives early such that the insider always has no less than σ_F^2/N unused information available in each period. Hence, he utilizes

σ_F^2/N private information, and if there is any left unused (i.e., if $\Sigma_n > 0$), the insider saves it for future trading through dissimulation (i.e., by adopting a mixed strategy).

Proposition 5. *If the inequalities in (17) hold strictly for $n \leq N-1$, the insider adopts a mixed strategy in all but the last period and the equilibrium in period n has the following properties*

$$\lambda_n = \frac{\sigma_F}{2\sqrt{N}\sigma_u}, \tag{18}$$

$$E[\pi_n] = \frac{\sigma_F \sigma_u}{2\sqrt{N}}, \tag{19}$$

$$U_n = (1 - n/N)\sigma_F^2, \tag{20}$$

where U_n, which negatively measures price informativeness, is the uncertainty of the asset liquidation value conditional on asset prices till period n:

$$U_n \equiv Var\left(F|P_1^*, ..., P_n^*\right).$$

The conditions in (17), with strict inequalities, guarantee that the insider's information arrives sufficiently in early rounds such that he always has more than σ_F^2/N private information in each period. As shown in Proposition 4, the insider utilizes σ_F^2/N private information each period. To achieve that, the insider needs to adopt a mixed strategy in all but the last period to dissimulate his private information. Since the insider utilizes information at a constant rate, his price impact and expected trading profit are also constant across period, as shown in equations (18) and (19), respectively. Finally, since the insider utilizes his private information at a constant rate, as shown in equation (20), the price informativeness increases linearly over time.

It is interesting to compare the above results with those in HHL (2001), where the insider receives all his private information in the first period: $\sigma_{F_1}^2 = \sigma_F^2$ and $\sigma_{F_i}^2 = 0$ for $i = 2, ...N$. This is a special case of (17). In equilibrium, the insider adopts a mixed strategy and utilizes the same amount of private information each period. Proposition 5 shows that these results hold more generally under the conditions in (17).

4.2 Examples: Monotonic Information Arrivals

In this subsection, we consider two examples with monotonic information arrival patterns. In Case 1, the insider's information arrives at a decreasing rate, that is

$$\sigma^2_{F_n} > \sigma^2_{F_{n+1}} \text{ for } n = 1, ..., N-1. \tag{21}$$

In Case 2, the insider's information arrives at an increasing rate, that is

$$\sigma^2_{F_n} < \sigma^2_{F_{n+1}} \text{ for } n = 1, ..., N-1. \tag{22}$$

These two cases are a generalized version of the two cases in the two-period example. In Case 1, more private information arrives at in early rounds. Since condition (21) is a special case of (17), as shown in Proposition 4, the insider adopts a mixed strategy in all but the last period and utilizes the same amount of information each period, $k_n^{2*} = \sigma_F^2/N$, for $n = 1, ..., N$. That is, the insider can can perfectly smooth his information usage over time.

Constant information usage is not feasible in Case 2. Since the private information arrives at an increasing rate, the insider does not possess enough private information in early rounds to utilize σ_F^2/N information each period. The equilibrium in Case 2 is summarized in the following proposition.

Proposition 6. *Under the conditions in (22), the insider adopts a pure strategy in every period and the equilibrium in period n, for $n = 1, ..., N$, has the following properties*

$$k_n^* = \sigma_{F_n}, \tag{23}$$

$$\lambda_n = \frac{\sigma_{F_n}}{2\sigma_u}, \tag{24}$$

$$E[\pi_n] = \frac{\sigma_{F_n}\sigma_u}{2}, \tag{25}$$

$$U_n > (1 - n/N)\sigma_F^2, \;\; n \neq N. \tag{26}$$

Anticipating the arrival of more private information in the future, the insider utilizes his current private information more aggressively. In fact, as shown in (23), the insider utilizes all his private information (i.e., adopts a pure strategy) each period. It has been noted in the literature that a monopolistic insider has the incentive to minimize the price impact by either breaking down his order into small ones (Kyle, 1985) or by adding noise to his order (HHL, 2001) to "go down" the market maker's demand curve. Proposition 6 shows that the expectation of future private information expedites the insider's usage of his private information. It generalizes the results in the two-period example and shows that when private information arrives at an increasing rate, the insider chooses to fully utilize his private information each period. Moreover, since the insider utilizes information at an increasing rate, his price impact and expected trading profits also increase over time, as shown in equations (24) and (25). Finally, since the insider's information budget constraint is binding, he utilizes less information than in Case 1 and the stock price informativeness is lower than that in Case 1 (equation (26)).

Numerical Example. To further illustrate the equilibrium in detail, we analyze a numerical example of Cases 1 and 2. Specifically, we set $N = 10$, $\sigma_F^2 = 1$, and $\sigma_u^2 = 0.1$. The insider's private information arrives at a linearly decreasing rate in Case 1:

$$\sigma_{F_n}^2 = \frac{2(N-n+1)}{N(N+1)}\sigma_F^2, \tag{27}$$

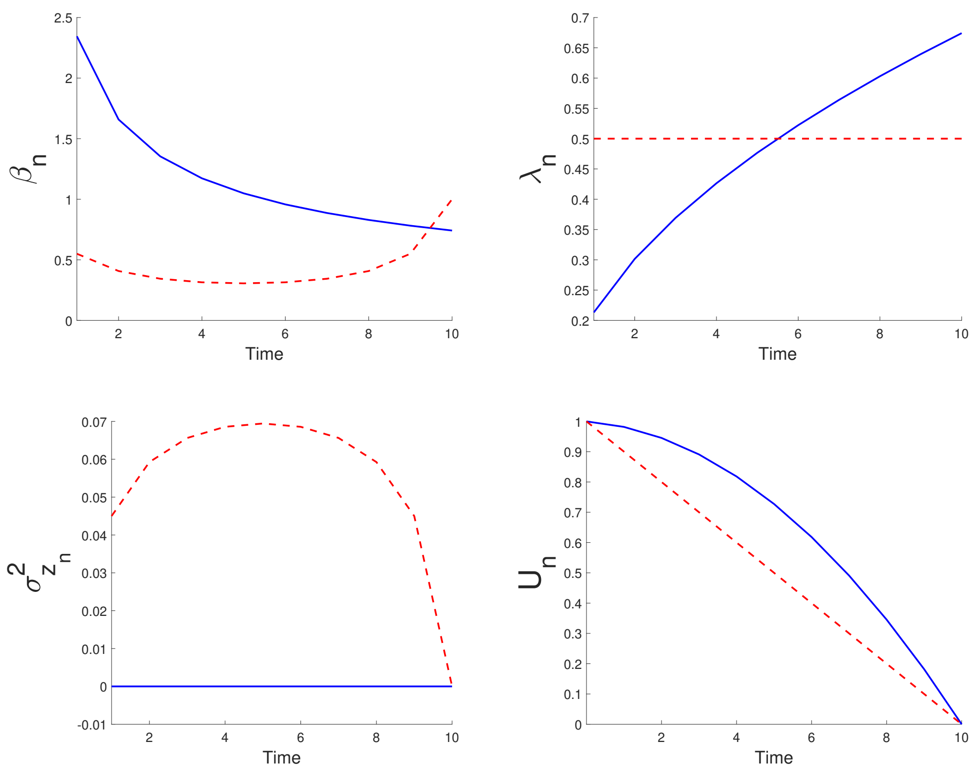
and at a linearly increasing rate in Case 2:

$$\sigma_{F_n}^2 = \frac{2n}{N(N+1)}\sigma_F^2. \tag{28}$$

The equilibria in these two cases are summarized in Figure 2. The upper left panel plots the trading intensity β_n against the trading period n. The dashed line represents Case 1, the case with a decreasing information arrival rate in (27), while the solid line represents Case 2, the case with an increasing information arrival rate in (28). In Case 1, anticipating less private information in later periods, the insider exploits his current private information less aggressively (β_n is smaller) in early periods to save his information for later periods. In contrast, when anticipating an increasing information arrival rate in Case 2, as shown by the solid line, the insider would exploit his private information more aggressively in earlier periods.

Figure 2. Equilibrium under Monotonic Information Arrivals

This figure plots the trading intensity β_n, price impact λ_n, the noise in the insider's demand $\sigma^2_{z_n}$, and price informativeness U_n respectively, for the case with a decreasing information arrival rate as specified in equation (27) (dashed line) and the case with an increasing information arrival rate as specified in equation (28) (solid line). Parameter values: $\sigma_F^2 = 1, \sigma_u^2 = 0.1$, and $N = 10$.



The upper right panel reports the price impact over time in equilibrium. As noted in Proposition 5, in Case 1 (represented by the dashed line), the insider utilizes the same amount of private information each period, leading to a constant price impact. The solid line shows that when private information arrives at increasing rate in Case 2, the price impact increases over time. This is because, as shown in Proposition 6, the insider utilizes all available private information each period and hence exploits more private information over time.

The lower left panel reports how the insider dissimulates his private information. The dashed line shows that in Case 1, the case with a decreasing information arrival rate, the insider adopts a mixed strategy (i.e., $\sigma_{z_n}^2 > 0$) in all but the last period. In contrast, when the insider's private information arrives at an increasing rate in Case 2, as shown by the solid line, he always adopts a pure strategy (i.e., $\sigma_{z_n}^2 = 0$).

Finally, the lower right panel plots the price informativeness measure U_n, which is the uncertainty about the liquidation value conditional on asset price history till period n, against time n. The dashed line shows that in the case with a decreasing information arrival rate, the insider utilizes the same amount of information each period. The uncertainty decreases linearly since the asset price reveals the same amount of information each period. In Case 2, where the insider's private information arrives at an increasing rate, the insider possesses less private information in earlier periods. Although all private information is revealed each period, the uncertainty still decreases more slowly than in the case with a decreasing information arrival rate (i.e., the solid line is above the dashed line).

5. Interpretation of the Mixed Strategy

When describing the mixed strategy played by an informed trader in Kyle-type models, researchers such as HHL (2001) and Yang and Zhu (2020) often loosely interpret it as the informed trader adding noise through randomization. This interpretation usually has a flavor that the trader consciously randomizes by actively choosing the amount of noise in his strategy.[2] However, in the game theory literature, it is well recognized that this interpretation,

[2]For instance, when defining dissimulation as the mixed strategy, HHL (2001, p. 666) state that "(t)he strategy balances immediate profits from informed trades against the reduction in future profits following trade disclosure and, hence, revelation of some of the insider's information. Our results show the optimality of adding a random noise component to informed trades, thereby diminishing the market maker's ability to draw inferences from the public record."

dubbed as a "naive" interpretation of "mixed strategies as objects of choices" by Osborne and Rubinstein (1994, p. 37), is not entirely satisfactory.[3] In a mixed-strategy equilibrium, the insider does not deliberately choose the noise component of his order. Instead, he simply passively takes the noise component as given. In our model, the insider is just indifferent across all orders given the market maker's pricing rules, and he is not actively choosing the volatility of the noise, σ_{z_n}. In this sense, the value of σ_{z_n} is determined by the market maker's equilibrium behavior, and the insider takes the value passively.

To further illustrate the subtle difference between the active choice and the passive determination, let us consider the classic game of "Battle of the Sexes." In this classic game, a man and a woman must decide on their weekend plans with the mutual preference of spending time together rather than apart. The man leans towards watching a movie, while the woman favors going shopping. If they opt for the movie together, the man's payoff is 2, and the woman's is 1; conversely, if they choose shopping, the payoffs switch to 1 for the man and 2 for the woman. If they elect to spend the weekend apart, both receive a payoff of 0. It is well-known that the unique mixed strategy equilibrium is that the man goes to watch a movie with probability 2/3 and the woman goes shopping with probability 2/3. Note that when we pin down the probability for one player, the man for instance, we do not analyze the man's behavior, but rather, we examine the indifference condition of the woman. In this sense, the man's mix probability is not the man's active choice, but passively determined by the woman's equilibrium behavior.

[3] When discussing mixed strategies, Rubinstein (1991, p. 912-913) wrote: "The concept of mixed strategy has often come under heavy fire. To quote Aumann (1987a): 'Mixed strategy equilibria have always been intuitively problematic... ', and Radner and Rosenthal (1982): 'One of the reasons why game-theoretic ideas have not found more widespread application is that randomization, which plays a major role in game theory, seems to have limited appeal in many practical situations.' The reason for the criticism is that the naive interpretation of a mixed strategy as an action which is conditional on the outcome of a lottery executed by the player before the game, goes against our intuition. We are reluctant to believe that our decisions are made at random. We prefer to be able to point to a reason for each action we take." The literature has suggested ways of interpreting mixed strategies based on purification, beliefs, large populations, and evolution (see the discussions in Osborne and Rubinstein (1994) and Oechssler (1997)).

To accommodate the usual interpretation of the insider intentionally randomizing, we consider an alternative game in which the insider can commit a linear trading strategy in each period as specified in equation (4) and then deliberately chooses all trading parameters $\{\beta_n \sigma_{z_n}\}_{n=1}^N$ at the beginning of the economy, say, in period 0 before any trading occurs.[4] The commitment is common knowledge in the game. Other features of the model remain the same as the baseline model presented in Section 2. In our context, such a committed trading strategy can be interpreted as a predetermined trading plan that specifies in advance the trading rule according to an algorithm. The equilibrium in this variation game is such that the insider chooses β_n and $\sigma_{z_n}^2$ to maximize his expected total trading profit over N periods and the market maker takes commitment (4) as given and sets the asset price according to his expected liquidation value of the risky asset.

Proposition 7. *The equilibrium in the variation game with commitment is identical to that characterized in Theorem 1.*

The above proposition shows that the mixed strategy analyzed in Section 3. can be thought of as the outcome of an optimization problem where the insider choose the optimal amount of noise in his demand to dissimulate his private information, which therefore formalizes the idea of "mixed strategies as objects of choice." Note that, in the variation game with commitment, in the worst case scenario, the insider can commit to the equilibrium trading strategy and hence earn the same expected profits as the insider in the baseline model. Hence, the commitment is expected to have a non-negative value in general. The equivalence between the equilibrium in this variation game and that in our baseline model is due to the fact that the commitment does not have value.

Our finding is consistent with the recent paper by Bernhardt and Boulatov (2023), who

[4]Under this specification, the insider chooses all trading parameters simultaneously before observing any information. Alternatively, we can also assume that the insider chooses trading parameters sequentially and after observing his private information and public information. For instance, in the two-period economy, the insider can choose $\{\beta_1, \sigma_{z_1}\}$ in period 1 after observing F_1 and choose $\{\beta_2, \sigma_{z_2}\}$ in period 2 after observing $\{P_1^*, F_1, F_2\}$. Our results are the same under this alternative assumption.

show that commitment has no value in a one-period Kyle model.[5] We analyze a multi-period setting with mixed strategies, and use the finding to interpret the mixed strategies in our baseline model as predetermined trading plans implemented by algorithms. Of note, Bernhardt and Boulatov (2023) also show that in games in which shocks are not normally distributed and so the equilibrium is nonlinear, commitment does have value.

6. Conclusion

We analyze a dynamic model of a monopolistic insider who receives private information on an ongoing basis and is subject to a post-trading disclosure requirement each period. We show that solving the equilibrium of this trading game is equivalent to solving the insider's optimal information usage problem, which is isomorphic to a standard consumption-saving problem where the agent has a power utility function and face a borrowing constraint. Hence, we can adopt the existing methods in the consumption-saving literature, such as dynamic programming, to solve for the equilibrium for our trading game.

Analogous to the "consumption-smoothing" intuition in the consumption-saving literature, the insider in our trading game "smooths" his information usage over time given the dynamic constraints imposed by the sequential arrival of his private information. Specifically, the dynamics of the insider's trade strategy are shaped by the comparison between his current private information and his anticipated future private information. Should the insider expects a reduction in his information advantage in the future, consistent with the insight in the existing literature, he would dissimulate his current private information through mixed strategies. Conversely, if the insider expects more information advantage in the future, he would not dissimulate and utilize all his private information by adopting a pure strategy.

Finally, we show that the mixed strategy in our model can be interpreted as the insider

[5] Bernhardt and Boulatov (2023) consider a Stackelberg setting in which the parameters chosen by the insider are observable to the market maker. Our result in Proposition 7 holds independent of whether the parameters of the insider's strategy are observable or not.

dissimulating his private information, i.e., actively choosing the size of the noise in his trading order to conceal his private information.

Appendix: Proofs

Proof of Theorem 1. The proof is by backward induction. We first claim that prior to the $(n+1)^{th}$ trade, the expected future profits have the following quadratic form in the linear equilibrium:

$$E(\sum_{i=n+1}^{N} \pi_i | F_1, ..., F_{n+1}, P_1^*, ..., P_n^*) = \alpha_n (\sum_{i=1}^{n+1} F_i - P_n^*)^2 + \delta_n, \tag{A.1}$$

where α_n and δ_n are constants with $\alpha_N = \delta_N = 0$. Then with the linear pricing functions (5) and (6) (or more generally, $P_n = P_{n-1}^* + \lambda_n y_n + f(y_1, ..., y_{n-1})$ and $P_n^* = P_{n-1}^* + \gamma_n x_n + g(y_1, ..., y_{n-1})$ where f and g are measurable functions and turn to be zero, as in Kyle's proof), moving backward by one step yields

$$\begin{aligned}
&E(\sum_{i=n}^{N} \pi_i | F_1, ..., F_n, P_1^*, ..., P_{n-1}^*) \\
&= E[x_n(F - P_n) + \alpha_n(\sum_{i=1}^{n+1} F_i - P_n^*)^2 + \delta_n | F_1, ..., F_n, P_1^*, ..., P_{n-1}^*] \\
&= x_n(\sum_{i=1}^{n} F_i - P_{n-1}^* - \lambda_n x_n) + \alpha_n(\sum_{i=1}^{n} F_i - P_{n-1}^* - \gamma_n x_n)^2 + \delta_n + \alpha_n \sigma_{F_{n+1}}^2.
\end{aligned} \tag{A.2}$$

Before proceeding with the maximization problem, we examine the semi-strong efficiency condition to get that,

$$\begin{aligned}
P_n &= E(F | P_1^*, ..., P_{n-1}^*, x_n + u_n) \\
&= P_{n-1}^* + E[\sum_{i=1}^{n} F_i - P_{n-1}^* | \beta_n(\sum_{i=1}^{n} F_i - P_{n-1}^*) + z_n + u_n] \\
&= P_{n-1}^* + \lambda_n(x_n + u_n)
\end{aligned}$$

with

$$\lambda_n = \frac{\beta_n(\Sigma_{n-1} + \sigma_{F_n}^2)}{\beta_n^2(\Sigma_{n-1} + \sigma_{F_n}^2) + \sigma_{z_n}^2 + \sigma_u^2}. \tag{A.3}$$

Analogously, $p_n^* = p_{n-1}^* + \gamma_n x_n$ with

$$\gamma_n = \frac{\beta_n(\Sigma_{n-1} + \sigma_{F_n}^2)}{\beta_n^2(\Sigma_{n-1} + \sigma_{F_n}^2) + \sigma_{z_n}^2}. \tag{A.4}$$

In deriving (A.3) and (A.4), we have used the relationship $E(\sum_{i=1}^{n} F_i - P_{n-1}^*)^2 = \Sigma_{n-1} + \sigma_{F_n}^2$ resulting from the independence between F_n and $\{F_1, ..., F_{n-1}, P_{n-1}^*\}$. In the following, results of Theorem 1 would be verified separately for *Case (i)* in which the insider employs a pure strategy and *Case (ii)* in which the insider employs a mixed strategy for the n^{th} trade.

Case (i). In the pure strategy case, $\sigma_{z_n}^2 = 0$. From the formula for the insider's trading strategy (4) and the market maker's pricing rule (2), we have

$$P_n^* = \sum_{i=1}^{n} F_i, \quad \Sigma_n = 0. \tag{A.5}$$

Consequently,

$$\sum_{i=1}^{n} F_i - P_{n-1}^* = P_n^* - P_{n-1}^* = \gamma_n x_n = \gamma_n \beta_n (\sum_{i=1}^{n} F_i - P_{n-1}^*)$$

from which, we obtain

$$\gamma_n = \frac{1}{\beta_n}. \tag{A.6}$$

In this case, we have

$$k_n^2 \equiv Var(P_n^* - P_{n-1}^*) = \Sigma_{n-1} + \sigma_{F_n}^2, \tag{A.7}$$

which, combined with (A.5) ensures that $\Sigma_n = \Sigma_{n-1} + \sigma_{F_n}^2 - k_n^2 = 0$. This means that (11) holds for n once it holds for $n-1$. In other words, pure strategy can ensure that (11) holds if the mixed strategy also ensures it (which will be shown to be the case shortly in Case (ii)).

From (A.5), the second term in (A.2) is zero, since $\sum_{i=1}^{n} F_i - P_{n-1}^* - \gamma_n x_n = \sum_{i=1}^{n} F_i - P_n^* = 0$, and thus the first-order condition (FOC) yields $x_n = \beta_n(\sum_{i=1}^{n} F_i - P_{n-1}^*)$ with

$$\beta_n = \frac{1}{2\lambda_n}. \tag{A.8}$$

The second-order condition (SOC) requires $\lambda_n \geq 0$ which is equivalent to $k_n \geq 0$ in equilibrium. With $\sigma_{z_n}^2 = 0$, from (A.3), (A.6), (A.7), and (A.8), we have

$$\lambda_n = \frac{\sqrt{\Sigma_{n-1} + \sigma_{F_n}^2}}{2\sigma_u} = \frac{k_n}{2\sigma_u},$$
$$\beta_n = \frac{\sigma_u}{\sqrt{\Sigma_{n-1} + \sigma_{F_n}^2}} = \frac{\sigma_u}{k_n},$$
$$\gamma_n = \frac{k_n}{\sigma_u}.$$

Finally, from these expressions, we can compute

$$E[\pi_n] = \beta_n(1 - \lambda_n\beta_n)(\Sigma_{n-1} + \sigma_{F_n}^2) = \frac{k_n}{2}\sigma_u.$$

Case (ii). In the mixed strategy case, $\sigma_{z_n}^2 > 0$. Note that we discuss this case only for $n < N$. The FOC of (A.2) gives

$$2(-\lambda_n + \alpha_n\gamma_n^2)x_n + (1 - 2\alpha_n\gamma_n)(\sum_{i=1}^{n} F_i - P_{n-1}^*) = 0.$$

Since this holds for all realizations of z_n in x_n, it requires

$$-\lambda_n + \alpha_n\gamma_n^2 = 0, \tag{A.9}$$

$$1 - 2\alpha_n\gamma_n = 0, \tag{A.10}$$

from which, we obtain

$$\gamma_n = 2\lambda_n. \tag{A.11}$$

From (A.3), (A.4) and (A.11),

$$\beta_n^2(\Sigma_{n-1} + \sigma_{F_n}^2) + \sigma_{z_n}^2 = \sigma_u^2, \tag{A.12}$$
$$\lambda_n = \frac{\beta_n(\Sigma_{n-1} + \sigma_{F_n}^2)}{2\sigma_u^2}. \tag{A.13}$$

In this case, we have

$$k_n^2 = \gamma_n^2 Var(x_n) = \gamma_n^2[\beta_n^2(\Sigma_{n-1} + \sigma_{F_n}^2) + \sigma_{z_n}^2] = \gamma_n^2\sigma_u^2$$

from which

$$\gamma_n = \frac{k_n}{\sigma_u}. \tag{A.14}$$

We have used the fact $\gamma_n \geq 0$ which is equivalent to $k_n \geq 0$ (note that if $k_n < 0$, then $\gamma_n = -k_n/\sigma_u$ and replacing k_n with $-k_n$ would not change anything). Indeed $\alpha_n \geq 0$ since otherwise, for some realized variables, the insider would get negative profits which would be dominated by not trading. So, from (A.10), $\gamma_n \geq 0$.

From (A.11) and (A.14),

$$\lambda_n = \frac{k_n}{2\sigma_u}. \tag{A.15}$$

Substituting (A.15) in (A.13) yields

$$\beta_n = \frac{k_n \sigma_u}{\Sigma_{n-1} + \sigma^2_{F_n}}. \tag{A.16}$$

Then substituting (A.16) in (A.12) can deliver the volatility of the random component in the order flow:

$$\sigma^2_{z_n} = (1 - \frac{k_n^2}{\Sigma_{n-1} + \sigma^2_{F_n}})\sigma^2_u. \tag{A.17}$$

We need to verify that $\sigma^2_{F_n}$ is positive in this case, i.e., $k_n^2 < \Sigma_{n-1} + \sigma^2_{F_n}$. Indeed,

$$\begin{aligned}
k_n^2 &= Var(P_n^* - P_{n-1}^*) \\
&< Var[(\sum_{i=1}^{n} F_i - P_n^*) + (P_n^* - P_{n-1}^*)] \\
&= Var(\sum_{i=1}^{n-1} F_i - P_{n-1}^*) + \sigma^2_{F_n} \\
&= \Sigma_{n-1} + \sigma^2_{F_n},
\end{aligned}$$

where the inequality follows from the fact that $(\sum\limits_{i=1}^{n} F_i - P_n^*)$ is independent of $P_n^* - P_{n-1}^*$ and that $Var(\sum\limits_{i=1}^{n} F_i - P_n^*) > 0$ when $\sigma^2_{z_n} > 0$ (since $Var(\sum\limits_{i=1}^{n} F_i - P_n^*) = Var(\sum\limits_{i=1}^{n} F_i - P_{n-1}^* | \beta_n(\sum\limits_{i=1}^{n} F_i - P_{n-1}^*) + z_n)$, positive when z_n is nondegenerate).

Now, from the projection theorem of normal variables, together with equations (A.12) and (A.17), we obtain

$$\begin{aligned}\Sigma_n &= Var(\sum_{i=1}^{n} F_i|P_1^*, ..., P_n^*, x_n)\\ &= Var[\sum_{i=1}^{n} F_i - P_{n-1}^*|\beta_n(\sum_{i=1}^{n} F_i - P_{n-1}^*) + z_n]\\ &= \frac{(\Sigma_{n-1} + \sigma_{F_n}^2)\sigma_{z_n}^2}{\sigma_u^2} = \Sigma_{n-1} + \sigma_{F_n}^2 - k_n^2,\end{aligned}$$

which verifies equation (11) for the mixed strategy case.

Furthermore, from (A.15), (A.16), and (A.17),

$$E[\pi_n] = \beta_n(1 - \lambda_n\beta_n)(\Sigma_{n-1} + \sigma_{F_n}^2) - \lambda_n\sigma_{z_n}^2 = \frac{k_n}{2}\sigma_u.$$

In this case, note that the insider strategy parameter β_n in equation (7) is well defined, since k_n is uniquely determined which would be shown later.

Finally, for both cases (i) and (ii), conjecture (A.1) can be justified by backward induction argument since when $n = N$, $\alpha_N = \delta_N = 0$ and when n is replaced by $n - 1$, it still holds, with recursions

$$\alpha_{n-1} = \beta_n(1 - \lambda_n\beta_n) + \alpha_n(1 - \gamma_n\beta_n)^2,\ \delta_{n-1} = \delta_n + \alpha_n\sigma_{F_{n+1}}^2 + \alpha_n\gamma_n^2\sigma_{z_n}^2. \tag{A.18}$$

In conclusion, all results given by equations (4)–(11) hold for both cases.

We now show how k_n is determined. First, from the pricing rule (1), what the insider obtains is what noise traders lose, that is

$$E[\pi_n] = \lambda_n\sigma_u^2 = \frac{k_n}{2}\sigma_u.$$

Hence, the maximization objective of insider's life-time profits in expectation can be written in reduced form as

$$\max_{\{k_1,\cdots,k_N\}\in\mathbb{R}_{\geq 0}^N} k_1 + \cdots + k_N \tag{A.19}$$

with budgets (13). The solution is unique from the convex optimization theory.

□

Proof of Proposition 1 . Results (15) and (16) and the fact that $\Sigma_n = 0 (> 0)$ corresponds to the pure (mixed) strategy have been shown in the proof of Theorem 1. Define the value function

$$V_n(\Sigma_{n-1}) = \max_{k_n^2 \le \Sigma_{n-1} + \sigma_{F_n}^2} k_n + k_{n+1}^* + \cdots + k_N^* \tag{A.20}$$

where k_t^* is the optimal corresponding in period t. According to the dynamic programming theory, $V_n(\Sigma_{n-1}) = \max_{k_n}[k_n + V_{n+1}(\Sigma_n)],$ for $n \le N$, with $V_{N+1} = 0$.

□

Proof of Proposition 2. We apply the recursive method in Proposition 1. Let $N = 2$. In period 2, with unused information scale Σ_1 and the information endowment $\sigma_{F_2}^2$, the insider's problem is:

$$\max_{k_2^2 \le \Sigma_1 + \sigma_{F_2}^2} k_2. \tag{A.21}$$

Solving this problem, we obtain $k_2^* = V_2(\Sigma_1) = \sqrt{\Sigma_1 + \sigma_{F_2}^2}$. Hence, from (16), $\Sigma_2 = \Sigma_1 + \sigma_{F_2}^2 - k_2^2 = 0$. According to Proposition 1, $\sigma_{z_2}^2 = 0$.

Now consider period 1. With information endowment $\sigma_{F_1}^2$, since $\Sigma_1 = \sigma_{F_1}^2 - k_1^2$, the insider's problem becomes

$$\max_{k_1^2 \le \sigma_{F_1}^2} k_1 + \sqrt{\sigma_{F_1}^2 + \sigma_{F_2}^2 - k_1^2}. \tag{A.22}$$

The optimal solution has two cases:

Case 1. If $\sigma_{F_1} > \sigma_{F_2}$, the optimum is $k_1 = \sigma_F/\sqrt{2}$. In this case, $\Sigma_1 = \sigma_{F_1}^2 - k_1^2 = (\sigma_{F_1}^2 - \sigma_{F_2}^2)/2 > 0$, which accroding to Proposition 1 means $\sigma_{z_1}^2 > 0$. Specifically, from (10), $\sigma_{z_1}^2 = \frac{\sigma_{F_1}^2 - \sigma_{F_2}^2}{2\sigma_{F_1}^2}\sigma_u^2$.

Case 2. If $\sigma_{F_1} \le \sigma_{F_2}$, the optimum is $k_1 = \sigma_{F_1}$. In this case, $\Sigma_1 = \sigma_{F_1}^2 - k_1^2 = 0$. Hence

$\sigma_{z_1}^2 = 0$ according to Proposition 1. Results about β, λ, and γ are direct from (7), (8), and (9) respectively. $\square$

Proof of Proposition 3. With conditions in (13), given the last constraint being equality, we can show

$$(k_1 - \overline{k})^2 + ... + (k_N - \overline{k})^2 = \sum_{i=1}^{N} k_i^2 - N\overline{k}^2 = \sigma_F^2 - \frac{(k_1 + ... + k_N)^2}{N},$$

from which we can observe that the maximization problem defined by (12) and (13) is equivalent to the minimization problem in Part (1) of Proposition 3. Since $\lambda_n = k_n/(2\sigma_u)$, problem in Part (2) is equivalent to problem in Part (1). $\square$

Proof of Proposition 4. The *necessity* is obvious. Now for *sufficiency*, from Theorem 1, we only need to show that $k_n^2 = \sigma_F^2/N$ is feasible for all $n \leq N$. Firstly, $k_1^2 = \sigma_F^2/N$ is feasible since the information available satisfies $\sigma_{F_1}^2 \geq \sigma_F^2/N$. In general, if strategies $k_{t-1}^2 = \sigma_F^2/N,\ t \leq n$ are all feasible and have been taken by the insider, then the feasible space for k_n^2 is $[0, \sigma_{F_n}^2 + \Sigma_{n-1}]$ with $\Sigma_{n-1} = \sum_{i=1}^{n-1}(\sigma_{F_i}^2 - \frac{\sigma_F^2}{N})$. The condition $\frac{n}{N}\sigma_F^2 \leq \sum_{i=1}^{n}\sigma_{F_i}^2$ is equivalent to $\frac{\sigma_F^2}{N} \leq \sigma_{F_n}^2 + \Sigma_{n-1}$ which precisely establishes the feasibility of $k_n^2 = \frac{\sigma_F^2}{N}$. $\square$

Proof of Proposition 5. Proposition 4 guarantees that in equilibrium $k_n = \frac{\sigma_F}{\sqrt{N}}$. With these strategies, $\Sigma_n = \sum_{i=1}^{n}(\sigma_{F_i}^2 - \sigma_F^2/N) > 0$ for $n \leq N-1$ and $\Sigma_N = 0$. From Proposition 1, insider adopts mixed strategies before the last period and pure strategy in the last period. Equations (18) and (19) follow directly. By definition, $U_n = \Sigma_n + Var(F_{n+1} + \cdots + F_N) = (1 - n/N)\sigma_F^2$. $\square$

Proof of Proposition 6. Suppose otherwise, if the insider does not always play pure strategies, then let us consider the first mixed one. Formally, denote

$$n_0 = inf\{n, \sigma_{z_n}^2 > 0\}.$$

Then from Theorem 1, $k_n^2 = \sigma_{F_n}^2$ and $\Sigma_n = 0$ for $n \leq n_0 - 1$ (if $n_0 = 1$, denote $k_0^2 = \sigma_{F_0}^2 = \Sigma_0 = 0$). Moreover, $k_{n_0}^2 < \sigma_{F_0}^2$ since $\sigma_{z_{n_0}}^2 > 0$ and $\Sigma_{n_0-1} = 0$. Thus there must exist some

$n_1 > n_0$, such that $k_{n_1}^2 > \sigma_{F_{n_1}}^2$ to ensure that $\sum\limits_{n=n_0}^{N} k_n^2 = \sum\limits_{n=n_0}^{N} \sigma_{F_n}^2$. Now claim that in this case, if k_{n_0} and k_{n_1} are replaced by $\sqrt{k_{n_0}^2 + \epsilon}$ and $\sqrt{k_{n_1}^2 - \epsilon}$ respectively, with ϵ positive and small enough, and with other k_n unchanged, then $\sum\limits_{n=n_0}^{N} k_n$ can be larger. In fact, since $k_{n_1}^2 > \sigma_{F_{n_1}}^2 \geq \sigma_{F_{n_0}}^2 > k_{n_0}^2$, we can let $\epsilon \in (0, k_{n_1}^2 - k_{n_0}^2)$. Then it is direct to show that

$$\sqrt{k_{n_0}^2 + \epsilon} + \sqrt{k_{n_1}^2 - \epsilon} > k_{n_0} + k_{n_1}.$$

This contradicts the maximization objective (12). Hence, insider always adopts pure strategies. From Proposition 1 and (16), $k_n^2 = \sigma_{F_n}^2$ always hold. Equations (23)-(25) follow directly. From $\Sigma_n = 0$ and that $\sigma_{F_n}^2$ increases with n, $U_n = \sigma_{F_{n+1}}^2 + \cdots + \sigma_{F_N}^2 > (1 - n/N)\sigma_F^2$, $n \neq N$. $\square$

Proof of Proposition 7. In this proof, we consider the 2-periods model, and the N-periods cases are similar.

Recall that before all trading begins, the insider commits to the following strategies in period-1 and 2, respectively,

$$x_i = \beta_i(F - P_{i-1}^*) + z_i,\ i = 1, 2, \tag{A.23}$$

The noise z_i is normally distributed and independent of all other variables. Trading parameters β_i and $Var(z_i)(\equiv \sigma_{z_i}^2))$ are decision variables chosen by the insider at the beginning of the economy. The insider's aim is to maximize the life-time profits in ex ante expectation:

$$\max_{\{\beta_i, \sigma_{z_i}^2\}_{i=1,2}} E[\pi_1 + \pi_2], \tag{A.24}$$

where $\pi_i = (F - P_i)(\beta_i(F - P_i^*) + z_i)$, for $i = 1, 2$.

With (A.23), market makers know that market orders are normally distributed with zero-mean and hence from projection theorem, they set pricing functions as

$$P_i = P_{i-1}^* + \lambda_i(x_i + u_i),\ i = 1, 2,\ and\ P_1^* = P_0^* + \gamma_1 x_1 \tag{A.25}$$

with

$$\lambda_i = \frac{\beta_i(\Sigma_{i-1} + \sigma_{F_i}^2)}{\beta_i^2(\Sigma_{i-1} + \sigma_{F_i}^2) + \sigma_{z_i}^2 + \sigma_u^2}, \ i = 1, 2, \ \textit{and} \ \gamma_1 = \frac{\beta_1(\Sigma_0 + \sigma_{F_1}^2)}{\beta_1^2(\Sigma_0 + \sigma_{F_1}^2) + \sigma_{z_1}^2}. \tag{A.26}$$

Now, we compute the profits in (A.24). With the committed trading strategies (A.23) and corresponding pricing functions (A.25), we can compute

$$E[\pi]_1 = E[(F - P_1)(\beta_1(F - P_0^*) + z_1)] = (1 - \lambda_1\beta_1)\beta_1\sigma_{F_1}^2 - \lambda_1\sigma_{z_1}^2, \tag{A.27}$$

and

$$E[\pi_2] = E[(F - P_2)(\beta_2(F - P_1^*) + z_2)] = (1 - \lambda_2\beta_2)\beta_2 E(F - P_1^*)^2 - \lambda_2\sigma_{z_2}^2, \tag{A.28}$$

where

$$E(F - P_1^*)^2 = \sigma_{F_2}^2 + E(F_1 - P_1^*)^2 = \sigma_{F_2}^2 + (1 - \gamma_1\beta_1)^2\sigma_{F_1}^2 + \gamma_1^2\sigma_{z_1}^2. \tag{A.29}$$

With (A.27), (A.28), and (A.29), we can express the insider's problem (A.24) as

$$\max_{\{\beta_i, \sigma_{z_i}^2\}_{i=1,2}} (1 - \lambda_1\beta_1)\beta_1\sigma_{F_1}^2 - \lambda_1\sigma_{z_1}^2 + (1 - \lambda_2\beta_2)\beta_2[\sigma_{F_2}^2 + (1 - \gamma_1\beta_1)^2\sigma_{F_1}^2 + \gamma_1^2\sigma_{z_1}^2] - \lambda_2\sigma_{z_2}^2$$

The maximization about $\sigma_{z_2}^2$ yields, $\sigma_{z_2}^2 = 0$, and the SOC is $\lambda_2 > 0$. The FOC about β_2 yields $\beta_2 = 1/(2\lambda_2)$. With these results, from (A.26),

$$\beta_2 = \frac{\sigma_u}{\sqrt{\Sigma_1 + \sigma_{F_2}^2}} \quad \textit{and} \quad \lambda_2 = \frac{\sqrt{\Sigma_1 + \sigma_{F_2}^2}}{2\sigma_u}. \tag{A.30}$$

The FOC about $\sigma_{z_1}^2$ yields

$$-\lambda_1 + \frac{\gamma_1^2\sigma_u}{2\sqrt{\Sigma_1 + \sigma_{F_2}^2}} = 0. \tag{A.31}$$

With (A.31), the FOC about β_1 gives

$$1 - \frac{\gamma_1\sigma_u}{\sqrt{\Sigma_1 + \sigma_{F_2}^2}} = 0. \tag{A.32}$$

All SOCs are satisfied. (A.31) and (A.32) together give the same result as in the main setup

$$\gamma_1 = 2\lambda_1.$$

These are key steps. Other results incuding the determination of k_n are the same as those in Theorem 1 and Proposition 1.

$\square$

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