Candidate Opinions Versus Voter Opinions

Joseph McMurray*

April 20, 2022

Abstract

When voters and candidates all possess private information about a common interest policy decision, a policy platform that is pivotal in an election, like a pivotal vote, can partially reveal others' private signals. This paper shows that this can dramatically alter candidate incentives, for example inducing a candidate with extreme liberal opinions to implement conservative policies, or vice versa. Multiple equilibria can also arise, which can be inefficient.

JEL Classification Number D72, D82

Keywords: Polarization, Jury Theorem, Common Interest, Pivotal, Strategic Voting, Epistemic Democracy

1 Introduction

One of the foundational models of elections is the common interest framework, pioneered by Condorcet (1785), in which voters and candidates value the same public outcomes (e.g. peace and prosperity) but each possess only limited information about which policy will best produce the desired outcomes. In such an environment, democracy can serve to distill truth from the myriad of opinions by pooling the collective knowledge of an entire electorate. As Condorcet's *jury theorem* points out, collective decisions may be highly accurate even when individual voter opinions are not.

In common interest environments, individuals can improve their actions by learning from others. In a seminal paper, Austen-Smith and Banks (1996) point out that voters can learn from one another even when communication is impossible, by appreciating the significance of a *pivotal event*, meaning that a single vote creates or

*Brigham Young University Economics Department. Email joseph.mcmurray@byu.edu.

breaks a tie. When a vote is pivotal, a voter can infer that other voters have voted in equal proportions for either of two policies or candidates. This is unlikely, but if it does not occur, a voter's own behavior does not influence his policy utility.[1] In equilibrium, therefore, voters strategically restrict attention to such an event.

Subsequent literature has highlighted several situations where the pivotal voting calculus dramatically alters voting behavior, sometimes in ways that seem somewhat pathological. In Feddersen and Pesendorfer (1996), pivotal voting considerations lead relatively uninformed voters to abstain from voting, wary that a pivotal vote likely contradicts the majority, thus overturning an informed decision. In Feddersen and Pesendorfer (1998), jurors may vote unanimously to convict a defendant that they each privately believe is innocent, expecting that their vote will only be pivotal if everyone else agrees. In Tajika (2019), each voter votes for the policy that seems *worst*, convinced that elections are closest when private information is misleading.[2]

In McMurray (2020), I point out that politicians, like voters, can make strategic inference from pivotal events. In that model, two candidates simultaneously choose platforms from a policy continuum, before voters vote for the candidate whose platform seems superior. If a candidate loses the election, her policy choice does not affect her utility, so the event of winning becomes a pivotal event, from which she can infer additional information from voters. This is important in that paper because candidates have no private signals of their own, and would therefore be inclined to adopt identical policies at the center of the policy interval. In equilibrium, however, the pivotal calculus leads candidates to polarize: one takes a liberal position, reasoning that she will only win if voters determine that the optimal policy truly lies on the left; the other takes a position on the right, knowing that if she wins it will be because the optimal policy is conservative.

The assumption in McMurray (2020) that candidates possess no private information is of course unrealistic: in reality, candidates have much better policy information than voters, because of privileged access to information and career incentives to learn about policy. Accordingly, this paper augments that model with candidate signals that may be more informative than voters'. I then evaluate whether, or to what extent, candidates whose information is superior to voters' still rely on pivotal infer-

[1] In this paper, masculine and feminine pronouns refer to voters and candidates, respectively.

[2] Outside the common interest literature, the pivotal voting calculus determines voters willingness to pay voting costs (Riker and Ordeshook, 1968) and to vote strategically for candidates who are less preferred but more likely to win (e.g. Myerson and Weber, 1993).

ence.

The most basic result of this is that candidates do still rely on pivotal inference: though a candidate's own signal might be far superior to a voter's, the pivotal calculus reveals information about voters' *collective* opinion—which, as the jury theorem points out, may be nearly infallible. In fact, this pivotal inference can be so strong that a candidate whose private opinion is extremely liberal implements an extremely conservative policy, or vice versa. In that sense, candidates rely more on voters' information than they rely on their own signals.

Pivotal inference is complicated because anytime a candidate deviates to a different policy position, her deviation triggers new voting behavior that reveals different information in the pivotal event of her winning the election. This makes a candidate's utility non-monotonic in her own policy choice, and can generate multiple equilibria. What matters is a candidate's position relative to her opponent's, and candidate signals exacerbate this by producing lots of locations where her opponent might locate.

Because candidates in different positions infer different information from the pivotal event of winning, candidates who are ex ante identical can behave very differently from one another in equilibrium. In one equilibrium below, for example, one candidate reacts strongly to her private signal, proposing policies ranging from the far left to the far right, while the other barely responds to her private signal, preferring to remain close to the political center.

A common manifestation of these endogenous differences between candidates is that candidates polarize in their reactions to the same signal realizations, with one adopting platform policies that are consistently left or right of her opponent's. In fact, polarization may be so pronounced that candidate A's response to the most conservative signal is to the left of candidate B's response to the most liberal signal. This level of polarization can be sub-optimal from a welfare perspective, but tends to generate higher welfare than other equilibrium configurations by guaranteeing a candidate on both sides of the policy interval, allowing voters to ensure that the policy outcome is not too far from the social optimum.

2 The Model

A game is played as follows. First, candidates A and B simultaneously choose policy platforms x_A and x_B within an interval $X = [-1, 1]$ of policy alternatives. Next, N voters simultaneously each vote for one of the two candidates, where N follows a Poisson distribution with mean n, as in Myerson (1998). The candidate $w \in \{A, B\}$ who receives more votes (breaking a tie, if necessary, with equal probability) wins the election and implements her platform x_w. Voters and candidates then each receive the following policy utility,

$$u(x_w, z) = -(x_w - z)^2 \tag{1}$$

which decreases quadratically in the distance from some policy $z \in Z = X$, which nature designates as being superior to any other policy.

The location of the optimal policy z is unknown, but follows a known prior distribution F with density f. Before the game, candidates and voters also observe private signals that are informative of z. Candidate signals s_j are drawn from a finite set $S \subseteq \mathbb{R}$, according to a distribution G with mass function g. Voter signals $\tilde{s}_i$ are drawn from a set $\tilde{S} \subseteq \mathbb{R}$ according to a distribution H with density h. Conditional on z, all voter and candidate signals are independent. That $G \neq H$ is not essential for the analysis below, but allows the possibility that candidates are better informed about policy issues than a typical voter is.[3] Assume that F, G, and H satisfy Conditions 1 through 3.[4]

Condition 1 (Symmetry) *$f(-z) = f(z)$, $g(-s|-z) = g(s|z)$, and $h(-\tilde{s}|-z) = h(\tilde{s}|z)$ for all $(s, \tilde{s}, z) \in S \times \tilde{S} \times Z$.*

Condition 2 (Log-concavity) *$\frac{d^2}{dz^2} \ln[f(z)\, g(s_A|z)\, g(s_B|z)] \leq 0$ for all $(s_A, s_B, z) \in S^2 \times Z$.*[5]

[3] This is reasonable in that candidates are selected on the basis of policy expertise, have career incentives to deepen their expertise, and have privileged access to some sources of information. A continuous H is convenient in that the set of voters who are indifferent between the two candidates has measure zero, so that equilibrium is in pure strategies and has a simple threshold structure.

[4] Examples of F, G, and H that satisfy Conditions 1 through 3 include the linear distributions used in Section 4, or distributions proportional to normal densities.

[5] This implies that $\frac{d^2}{dz^2} \ln[f(z)\, g(s_A|z)\, g(s_B|z)] \leq 0$ and $\frac{d^2}{dz^2} \ln[f(z)\, h(s_j|z)] \leq 0$ for $j = A, B$ (An, 1998), and therefore that $\frac{d^2}{dz^2} \ln[f(z|s_A, s_B)] \leq 0$ and $\frac{d^2}{dz^2} \ln[f(z|s_j)] \leq 0$, as well.

Condition 3 (strict MLRP) *If* $s < s'$, $\tilde{s} < \tilde{s}'$, *and* $z < z'$ *then* $\frac{g(s'|z)}{g(s|z)} \leq \frac{g(s'|z')}{g(s|z')}$ *and* $\frac{h(\tilde{s}'|z)}{h(\tilde{s}|z)} \leq \frac{h(\tilde{s}'|z')}{h(\tilde{s}|z')}$.

Condition 1 states that symmetric states of the world are equally likely and produce symmetric distributions of candidate and voter signals. This does not seem essential for the results below, but simplifies the analysis substantially by reducing the number of cases that need to be analyzed. It also ensures that endogenous differences between candidates are not due to exogenous asymmetries in the policy environment. Condition 2 states that the joint density of z and s_A and s_B is log-concave in z. As Bagnoli and Bergstrom (2005) explain, this ensures that the expectation of z (both unconditionally and conditional on s_A and s_B) shifts monotonically as its domain is truncated. Condition 3 states that candidate and voter signals are informative of z in the sense of the strict monotone likelihood ratio property: higher signal realizations are more likely in higher states. As in McMurray (2017), private signals then reflect voter *ideology*: those with low signal realizations believe policies on the political left to be optimal, while those with high signals favor policies on th right.

Let Y denote the set of candidate strategies $y : S \rightarrow X$, which designate a policy platform in X for every signal realization $s \in S$. In the subgame associated with any pair (x_A, x_B) of candidate platorms, let V denote the set of voting strategies $v : \tilde{S} \rightarrow \{A, B\}$ which specify a vote choice in $\{A, B\}$ for every signal realization $\tilde{s} \in \tilde{S}$. A strategy v_t is said to be *ideological* if $v_t(\tilde{s}) = \begin{cases} \arg\min_j x_j & \text{if } \tilde{s} < t \\ \arg\max_j x_j & \text{if } \tilde{s} > t \end{cases}$, meaning that voters with ideologies sufficiently far left vote for the more liberal candidate while those with ideologies on the right vote for the more conservative candidate. In the complete game, let Σ denote the set of strategies $\sigma : X^2 \rightarrow V$ that specify a subgame strategy for every candidate pair $(x_A, x_B) \in X^2$.

Observing that $y_A(s_A) = x_A$ and $y_B(s_B) = x_B$ can give voters information about s_A and s_B, and therefore about z. If y_j is strictly increasing, for example, then $y_j(s_j)$ completely reveals s_j. Let $S_j(x_j) = \{s \in S : y_j(s_j) = x_j\}$ denote the support in S for policy x_j. If his peers all follow $v \in V$ in the voting subgame associated with (x_A, x_B), a voter's best response v^{br} therefore maximizes the expectation of (1) (over N, z, $(s_j)_{j=A,B}$, and $(s_i)_{i=1}^{N}$, which, together with the voting strategy v, determine the election winner w), conditional on his own private signal $\tilde{s}_i$ and on $S_A(x_A)$ and $S_B(x_B)$. $v^* \in V$ is a Bayes-Nash equilibrium (BNE) in the voting subgame if it is its own best response. In the broader game, (σ^*, y_A^*, y_B^*) is a perfect Bayesian equilibrium (PBE) if $\sigma^*(x_A, x_B)$ is a BNE for every subgame and $y_j^*(s)$ maximizes the expectation of (1) (over N, z, s_{-j}, and $(s_i)_{i=1}^N$) for each candidate $j \in \{A, B\}$, conditional on her private signal $s_j = s$.

3 Equilibrium Analysis

3.1 Voting

The analysis of voting behavior follows McMurray (2020) almost exactly. With quadratic utility, a voter prefers policies as close as possible to his expectation of z, conditional on available information. Voting A and voting B produce the same expected utility, however, unless a voter happens to be *pivotal* (event P) by making or breaking a tie, which is more likely in some states of the world than others. Best response voting takes event P into account, along with the private signal $\tilde{s}_i$. Since candidate platforms (fully or partially) reveal candidate information, a voter now also takes into account that $s_A \in S_A(x_A)$ and $s_B \in S_B(x_B)$.

Despite this additional information, Condition 3 guarantees that the expectation of z increases with $\tilde{s}_i$, so a voter with sufficiently liberal or sufficiently conservative ideology votes for the more liberal or the more conservative of the two candidates, respectively. In other words, best response voting is ideological, as Proposition 1 now states. The second part of Proposition 1 states that if $x_A \neq x_B$ then a unique equilibrium threshold exists, which increases in x_A and x_B.

Proposition 1 *1. $v^{br} \in V$ is a best response to $v \in V$ only if it is ideological, with threshold $t^{br} \in \tilde{S}$ such that $E\left[z|P, \tilde{s}_i = t^{br}, S_A(x_A), S_B(x_B)\right] = \frac{x_A+x_B}{2}$.*

2. There exists a unique $t^* : X \to \tilde{S}$ such that $v_{t^*(\bar{x})}$ with $\bar{x} = \frac{x_A+x_B}{2}$ is a BNE in the subgame associated with (x_A, x_B), and is the only BNE if $x_A \neq x_B$. $v_{t^*(\bar{x})}$ is also socially optimal in V and is the only social optimal strategy if $x_A \neq x_B$. Moreover, t^* increases in $\bar{x}$, and therefore in x_A and x_B.

In McMurray (2020) I show that, in large elections, the optimal voting strategy identifies the candidate $\arg\max_j u(x_j, z)$ whose platform is superior. Additional information conveyed by candidates' platforms in $S_A(x_A)$ and $S_B(x_B)$ only improves voters' ability to identify the superior candidate, so here, too, optimal voting is perfectly informative in the limit. Since voters behave optimally in equilibrium, Condorcet's jury theorem (stated here as a lemma) holds, and the superior candidate wins with probability approaching one in large elections.

Lemma 1 (Jury theorem) *If $x_A \neq x_B$ and $w_n^* \in \{A, B\}$ denotes the election winner when voting follows the unique BNE v_n^* for every n then $w_n^* \rightarrow_{a.s.} \arg\max_j u(x_j, z)$ as $n \rightarrow \infty$.*

3.2 Candidates

Let σ^* denote the voting strategy that induces $v_{t^*(\bar{x})}$ in every subgame. A perfect Bayesian equilibrium in the complete game requires this voting strategy, combined with optimal candidate platform strategy functions $y_j : S \rightarrow X$. X is compact and expected utility is bounded and continuous, so Theorem 3.1 of Balder (1988) guarantees the existence of an equilibrium pair $\left(y_{A,n}^*, y_{B,n}^*\right)$ of platform functions for any n. Proposition 2 now states that, in large elections, a candidate's equilibrium policy position is simply her expectation of the optimal policy. As in McMurray (2020), this expectation conditions on the event $w = j$ (or simply, event j) of winning the election, which is a candidate's way of being pivotal for the outcome. Here, however, her policy position also depends on her private signal s_j.[6]

Proposition 2 *If $\left(\sigma_n^*, y_{A,n}^*, y_{B,n}^*\right)$ is a sequence of PBE approaching $\left(\sigma_\infty^*, y_{A,\infty}^*, y_{B,\infty}^*\right)$ then:*

1. $\lim\limits_{n\rightarrow\infty} \left[y_{j,n}^*(s_j) - E(z|j, s_j; n)\right] = 0$ for all $s_j \in S$ and for $j = A, B$.

2. $y_{A,\infty}^*(s) \neq y_{B,\infty}^*(s')$ for all $s, s' \in S$.

3. If $s' < s''$ and $y_{A,\infty}^*(s) \notin \left[y_{B,\infty}^*(s'), y_{B,\infty}^*(s'')\right]$ for all $s \in S$ then $y_{B,\infty}^*(s') < y_{B,\infty}^*(s'')$.

Since candidates are identical, it might seem intuitive that they should respond identically to identical signals. If they did not account for the "pivotal" event of winning the election, for example, both would adopt the same policy function

[6] This simple addition of private signals substantially complicates the proof in McMurray (2019): adjusting a candidate's platform directly impacts her utility in the event of winning, but also affects the distribution of who wins; when candidates do not have private signals of their own, the latter impact is exactly offset by the response from voters, who share the candidate's preferences exactly when she has no private information. Additionally, with no candidate signals, each candidate chooses a single policy platform; here, each may choose a different policy platform for every signal realization, and must therefore anticipate the entire distribution of policy positions that her opponent might take.

$y_{A,n}(s) = y_{B,n}(s) = E(z|s)$ which, given the MLRP assumption, would be increasing in s. According to Part 2 of Proposition 2, however, candidates cannot adopt the same policy function in equilibrium when n is large. In fact, $y_{A,\infty}$ and $y_{B,\infty}$ must have disjoint domains.

To understand this result, first consider candidate B's best response when A plays a degenerate strategy $y_A(s) = x_A$ for all s. If B responds with a policy $x_B > x_A$ then, by the jury theorem, she will win if $z > \bar{x}$ and lose otherwise. Conditional on winning, therefore, her expectation updates to $\lim_{n\to\infty} E(z|B, s_B) = E(z|s_B, z > \bar{x})$. If x_B is sufficiently close to x_A then it is not a best response because $E\left(z|s_B, z > \frac{x_A+x_B}{2}\right) > x_A \approx x_B$. If x_B is sufficiently close to 1 then it is not a best response because $E\left(z|s_B, z > \frac{x_A+x_B}{2}\right) < 1 \approx x_B$. Between x_A and 1, $E(z|s_B, z > \bar{x})$ increases in x_B, but with derivative less than one (given the log-concavity of f), so a unique $x_{B,\infty}^{brR} \in (x_A, 1)$ gives a local best response in the limit, and limit utility increases on $\left[x_A, x_{B,\infty}^{brR}\right]$ and decreases on $\left[x_{B,\infty}^{brR}, 1\right]$. By symmetric reasoning, there is a local best response $x_B^{brL} \in (-1, x_A)$ such that utility increases on $\left[-1, x_B^{brL}\right]$ and decreases on $\left[x_B^{brL}, x_A\right]$. Expected utility is thus "M-shaped" in x_B, first increasing then decreasing then increasing then decreasing; $x_B = x_A$ locally *minimizes* expected utility and is never a best response. That policies just below x_B are "too high" while policies just above x_B are "too low" highlights an unusual feature of this environment, which is that a candidate's preference over policies depends on her interpretation of the event of winning an election, which depends on voter behavior, which reacts to her own platform choice.

If $y_A(s) = x_{A1}$ for all $s \in S$ then candidate B's limiting expected utility is locally minimized at x_{A1} and concave and single-peaked to the left and to the right of x_{A1}; if $y_A(s) = x_{A2}$ for all $s \in S$ then B's limiting expected utility is locally minimized at x_{A2} and concave and single-peaked to the left and to the right of x_{A2}. If the image of $y_{A,n}$ is the set $\{x_{A1}, x_{A2}\}$ then B's limiting expected utility is some weighted average of $\lim_{n\to\infty} E[u(x_w, z)|B, s_B; x_{A1}, x_B]$ and $\lim_{n\to\infty} E[u(x_w, z)|B, s_B; x_{A2}, x_B]$. As the limit of marginal expected utility increases discretely both at $x_B = x_{A1}$ and at $x_B = x_{A2}$, neither maximizes expected utility. Similarly, if $y_A(s)$ differs for every s (as it would if A implemented an increasing strategy such as $y_A(s) = E(z|s)$) then $x_B = y_A(s)$ never maximizes candidate B's limiting expected utility.

Though limiting marginal utility jumps discretely upward at every $y_A(s)$, it is continuous in intervals that do not contain $y_A(s)$ for any $s \in S$. In such intervals, the monotone likelihood ratio property guarantees that candidate B's expectation $\lim_{n\to\infty} E_{s_A,z}(z|B, s_B; y_A, x_B)$ of the optimal policy increases in s_B, so Part 3 of Lemma 2 states that $y^*_{B,\infty}(s)$ is increasing in such an interval. This rules out the possibility of pooling equilibria, wherein a candidate adopts the same policy in response to distinct signal realizations, and thus implies that equilibrium policy platforms perfectly reveal candidates' private information to voters. Ultimately, however, this does not influence voting in the limit, which aggregates information perfectly with or without the addition of candidates' private information.

The impact highlighted in Proposition 2 of the pivotal calculus on candidate behavior can be substantial. Suppose, for example, that a candidate's private signal heavily favors policies left of center, but that she nevertheless adopts a policy position right of center. If she wins, she infers that $z > \bar{x}$, despite her private signal to the contrary. Even when her signal is highly informative, so that it skews her posterior beliefs about z substantially by itself, the event of winning truncates her posterior distribution, effectively skewing it in the opposite direction.

When candidates do not observe private signals, I show in McMurray (2020) that the pivotal calculus has a polarizing effect on candidates: the best response to any platform is always on the opposite side of the policy spectrum. By extension, if candidate A's strategy mixes over multiple positions left of center, candidate B's best response should be the expectation over a set of policies that are all right of center. Given the weight of pivotal considerations, as explained above, it seems reasonable to conjecture that such polarizing forces might dominate a candidate's private signal. At this level of generality, however, it is difficult to rule out other possibilities. To make progress, Section 4 turns to numerical analysis.

4 Numerical Analysis

As a simple example, let z be uniform on X and let $S = \{s_1, s_2, ..., s_K\}$ where $s_k = -1 + 2\frac{k-1}{K-1}$, so that s_1 through s_K are spaced evenly between -1 and 1. For ease of computation, let $g(s|z)$ be proportional to $(1 + sz)$, and thus linear in both s and z. Section 4.1 analyzes the case of $K = 2$ and Section 4.2 treats larger K.

4.1 Two Signals

If $K = 2$ then $S = \{s_1, s_2\} = \{-1, 1\}$ and $g(s|z) = \frac{1}{2}(1 + sz)$. If candidate A adopts platforms $y_A(s_1) < y_A(s_2)$ then this partitions Z into three segments, and candidate B can respond with a policy in any of the three. If she adopts a platform to the left of $y_A(s_1)$ then, in a large election, she wins either if A observes s_1 and $z < \frac{y_A(s_1)+x_B}{2}$ or if A observes s_2 and $z < \frac{y_A(s_2)+x_B}{2}$. Her conditional expectation is then as follows, for $s \in S$.

$$\begin{aligned} &\lim_{n\to\infty} E(z|B, s; x_B < y_A(s_1) < y_A(s_2)) \\ &= \frac{\int_{-1}^{\frac{y_A(s_1)+x_B}{2}} zg(s_1|z) f(z|s)\, dz + \int_{-1}^{\frac{y_A(s_2)+x_B}{2}} zg(s_2|z) f(z|s)\, dz}{\int_{-1}^{\frac{y_A(s_1)+x_B}{2}} g(s_1|z) f(z|s)\, dz + \int_{-1}^{\frac{y_A(s_2)+x_B}{2}} g(s_2|z) f(z|s)\, dz} \end{aligned} \tag{2}$$

If she instead responds with a policy between $y_A(s_1)$ and $y_A(s_2)$ then she wins if A observes s_1 and $z > \frac{y_A(s_1)+y_B(s_k)}{2}$ or if A observes s_2 and $z < \frac{y_A(s_2)+y_B(s_k)}{2}$, so her conditional expectation is as follows.

$$\begin{aligned} &\lim_{n\to\infty} E(z|B, s; y_A(s_1) < x_B < y_A(s_2)) \\ &= \frac{\int_{\frac{y_A(s_1)+x_B}{2}}^{1} zg(s_1|z) f(z|s)\, dz + \int_{-1}^{\frac{y_A(s_2)+x_B}{2}} zg(s_2|z) f(z|s)\, dz}{\int_{\frac{y_A(s_1)+x_B}{2}}^{1} g(s_1|z) f(z|s)\, dz + \int_{-1}^{\frac{y_A(s_2)+x_B}{2}} g(s_2|z) f(z|s)\, dz} \end{aligned} \tag{3}$$

If she adopts a policy to the right of $y_A(s_2)$ then she will win the election if A observes s_1 and $z > \frac{y_A(s_1)+x_B}{2}$ or if A observes s_2 and $z > \frac{y_A(s_2)+x_B}{2}$. Conditional on winning, therefore, her expectation is given by the following.

$$\begin{aligned} &\lim_{n\to\infty} E(z|B, s; y_A(s_1) < y_A(s_2) < x_B) \\ &= \frac{\int_{\frac{y_A(s_1)+x_B}{2}}^{1} zg(s_1|z) f(z|s)\, dz + \int_{\frac{y_A(s_2)+x_B}{2}}^{1} zg(s_2|z) f(z|s)\, dz}{\int_{\frac{y_A(s_1)+x_B}{2}}^{1} g(s_1|z) f(z|s)\, dz + \int_{\frac{y_A(s_2)+x_B}{2}}^{1} g(s_2|z) f(z|s)\, dz} \end{aligned} \tag{4}$$

If $y_A(s_1) > y_A(s_2)$ then these three expectations can be rewritten with $y_A(s_1)$ and $y_A(s_2)$ reversed. Up to symmetry of the candidates and policy interval, there are three possible equilibrium platform configurations, which can be derived numerically. These are described below.

4.1.1 Equilibrium 1: polarization

If $y_A(s_1) < y_A(s_2) < y_B(s_1) < y_B(s_2)$ then the candidates are polarized, in the sense that $\max_{s \in S} y_A(s) < \min_{s \in S} y_B(s)$. Platforms $y_A(s)$ and $y_B(s)$ also both increase in s. An equilibrium of this type can be derived by equating (4) to $y_B(s)$ simultaneously for $s = s_1$ and for $s = s_2$, while also solving analogous equations for candidate A. This yields a unique solution $y^*_{A,\infty} \approx (-.59, -.27)$, $y^*_{B,\infty} \approx (.27, .59)$, which is symmetric around the origin.[7]

If a candidate based her expectation on her signal alone, observing private signal $s_2 = 1$ would lead her to form expectation $E(z|s_2) = \frac{1}{3}$. If she adopted $x_B = \frac{1}{3}$ as her platform when $y_A = (-.59, -.27)$, however, she would only win the election if it turned out that her opponent had observed a negative signal and $z > \frac{-.59+1/3}{2}$, or that her opponent had observed a positive signal (like she herself did) and $z > \frac{-.27+1/3}{2}$. Averaging over these possibilities, her new expectation would be higher than before: $\lim_{n \to \infty} E(z|B, s_2 = 1) \approx .54$. If she increased her platform to match this higher expectation, however, the circumstances in which she wins the election would be skewed even further in the same direction, so that her revised expectation is higher still ($\lim_{n \to \infty} E(z|B, s_2 = 1) \approx .58$). In equilibrium, she adopts $y^*_{B,\infty}(s_2) \approx .59$.

The pivotal inference that occurs when candidate B observes $s_1 = -1$ is similar, but is worth emphasizing separately. After her private signal alone, her expectation $E(z|s_1) = -\frac{1}{3}$ is negative. If she adopted $x_B = -\frac{1}{3}$ as her platform, however, she would only win the election if it turned out that she and her opponent had both observed negative signals and z exceeded $-.46$ (and so was closer to $-\frac{1}{3}$ than to $-.59$) or that their signals were opposite and z was smaller than $-.3$ (and thus closer to $-\frac{1}{3}$ than to $-.27$). The first of these possibilities includes all positive values of z, so averaging over these possibilities, her new expectation would be less negative: $\lim_{n \to \infty} E(z|B, s_2 = 1) \approx -.22$. If she adjusted her platform to $-.22$, however, she would now win for higher values of z, and her limiting expectation would increase to .10. Continually raising her platform would continually inflate her expectation, until she reached the best response position $x_B = .27$, where her limiting expectation would finally coincide with her policy position. By incorporating the pivotal calculus, then,

[7] The local concavity of expected utility only guarantees these to be local maxima; establishing this as an equilibrium also requires confirming numerically that neither candidate can generate higher utility by deviating to a different region of the policy space. The same is true of all of the equilibria derived in this section.

her eqilibrium expectation has the *opposite* sign from her original private expectation, and is almost as extreme. Candidates A and B respond to the same signal with different policies that are a distance .86 apart, and candidate B's equilibrium platform lies further to the right than A's, even when B observes $s_B = -1$ and A observes $s_A = 1$.

4.1.2 Equilibrium 2: partial polarization

If $y_A(s_1) < y_B(s_1) < y_A(s_2) < y_B(s_2)$ then candidates' platforms are partially polarized, in that $y_A(s_A) \leq y_B(s_B)$ for three of the four signal pairs (s_A, s_B) (with strict inequality when $s_A = s_B$). Platforms are no longer fully polarized, however, as $y_A(s_A) > y_B(s_B)$ is also possible. An equilibrium of this type can be derived by equating (4) to $y_B(s_2)$ and (3) to $y_B(s_1)$, along with analogous equations for candidate A. This yields a unique solution $y^*_{A,\infty} = (-.64, .19)$, $y^*_{B,\infty} = (-.19, .64)$, which is again symmetric around the origin.

In this equilibrium, candidate B reasons that if she observes s_2 and wins from a policy position .64, it will be either because the two candidates drew opposite signals but z is right of the origin, or both candidates drew positive signals but z is closer to .64 than to .19. Across all such possibilities, the average value of z is .64. On the other hand, if she observes s_1 and wins from a policy position $-.19$, it will be either because the candidates drew opposite signals but z is negative, or both candidates drew negative signals but z is closer to $-.19$ than to $-.64$. Across all such scenarios, the average value of z is $-.19$. Polarization here is incomplete, but still quite pronounced: candidates A and B respond to identical signals with platforms that differ by .45.

4.1.3 Equilibrium 3: centrist and extremist

If $y_A(s_1) < y_B(s_1) < y_B(s_2) < y_A(s_2)$ then both candidates' strategies increase in s, but candidate A adopts an extreme left or an extreme right position, while B adopts a moderate left or a moderate right position. An equilibrium of this type can be derived by equating (3) to $y_B(s)$ both for $s = s_1$ and for $s = s_2$, and using expressions analogous to (2) and (4) to solve $\lim_{n\to\infty} E(z|A, s_1; x_A < y_B(s_1) < y_B(s_2)) = y_A(s_1)$ and $\lim_{n\to\infty} E(z|A, s_2; y_B(s_1) < y_B(s_2) < x_A) = y_A(s_2)$. This yields a unique solution $y^*_{A,\infty} = (-.71, .71)$, $y^*_{B,\infty} = (-.15, .15)$, in which both candidates' strategies are symmetric around the origin.

Knowing that B takes only moderate positions, candidate A reasons that if she wins from $y_A = -.71$, it will be because z is low—closer to $-.71$ than to $-.15$ if $s_B = -1$, or than to .15 if $s_B = 1$. Across all such scenarios, the average value of z is $-.71$. On the other hand, if she wins from $y_A = .71$ it will be because z is high—in expectation equalling $\lim_{n\to\infty} E(z|B, s_2 = 1) \approx .71$. Thus, candidate A responds to a negative signal with a very low platform, and to a positive signal with a very high platform.

Based on her private signal alone, candidate B expects the optimal policy to be positive, with expectation $\lim_{n\to\infty} E(z|s_2 = 1) = \frac{1}{3}$. However, candidate B will only win the election if candidate A's platform is not already close to z. This can happen when z is close to zero, or when candidate A receives a signal with the opposite sign from z. When z is positive and s_A is negative, adjusting y_B has little utility consequence because she is already on the side of truth, so her quadratic utility increases little when she moves marginally toward z. When z is negative and s_A is positive, however, the incentive to move left is strong. Averaging across these possibilities, B formulates a limiting expectation $\lim_{n\to\infty} E(z|B, s_2 = 1) \approx .20$ slightly lower than her private expectation. As she adjusts her policy platform in that direction, her expectation falls further, until in equilibrium $y^*_{B,\infty}(s_2) \approx .15$.

4.1.4 No other equilibria

Since candidates are ex ante identical, there are of course limiting equilibria symmetric to the three above, with A and B trading roles. Otherwise, however, no other behavior can prevail as the limit of equilibria. In total, there are twelve possible orderings of $y_A(s_1)$, $y_A(s_2)$, $y_B(s_1)$, and $y_B(s_2)$ with A adopting the left-most position. Five of these specify that $y_A(s)$ or $y_B(s)$ decreases over an interval that does not include any opponent platform positions, which Proposition 2 rules out in equilibrium.

If it existed, an equilibrium with $y_A(s_2) < y_B(s_2) < y_A(s_1) < y_B(s_1)$ could be derived by solving the first-order equations that equate (3) to $y_B(s_2)$ and (4) to $y_B(s_1)$ and using expressions analogous to (2) and (3) to solve $\lim_{n\to\infty} E(z|A, s_2; y_B(s_2) < y_A(s_1) < y_B(s_1)) = y_A(s_1)$ and $\lim_{n\to\infty} E(z|A, s_2; y_A(s_2) < y_B(s_2) < y_B(s_1)) = y_A(s_2)$. However, this yields no numerical solutions. First-order conditions supporting $y_A(s_1) < y_B(s_2) < y_A(s_2) < y_B(s_1)$ (or $y_A(s_2) < y_B(s_1) < y_A(s_1) < y_B(s_2)$, which is equivalent up to a reversal of the policy space) and $y_A(s_2) < y_B(s_1) < y_B(s_2) < y_A(s_1)$ admit unique solutions, but the implied responses to opponent platforms maximize expected utility only locally, not globally.[8]

4.1.5 Welfare

The equilibria identified in this section underscores the importance of the pivotal calculus, as candidates who would otherwise behave identically now behave very differently from each other. In equilibria 1 and 2, candidates polarize substantially. In equilibrium 3, their behavior differs categorically, with A reacting strongly to her signal (i.e. taking a very low position when s_A is negative and a very high position when s_A is positive) while B's response is much more muted.

With multiple equilibria, it is natural to ask which equilibrium is most likely to prevail. In equilibria 1 and 2, candidates behave symmetrically; in equilibrium 3, each candidate responds symmetrically to symmetric signals. Regardless of which criteria one establishes for selecting an equilibrium, it is worth pointing out that equilibrium multiplicity is itself another result of the pivotal calculus: if they conditioned on their private information alone, candidates would behave identically, producing a unique equilibrium prediction.

Another natural question is which of the three equilibria produces the greatest welfare. Since voters and candidates share a common preference in this model, it is uncontroversial to reinterpret expected utility as social welfare. Since u is quadratic, the key for welfare is to ensure that the policy outcome is not too far from the optimum. The jury theorem guarantees that voters will identify the better of two platforms, so welfare will be high as long as scenarios are avoided where neither platform is close to z.

There does not seem to be an obvious intuition for which equilibrium should maximize welfare. Equilibrium 1 ensures policy options on both sides of the origin; equilibrium 2 extends further into the extremes of the policy space, and equilibrium

[8] $y_A \approx (-.65, .17)$ and $y_B \approx (.29, .10)$ solve the first set of first-order conditions, but $y_B(s_2) > .17$ brings candidate B higher utility. $y_A = (.44, -.44)$ and $y_B = (-.36, .36)$ satisfy the second set of first-order conditions, but $y_A(s_1) < .36$ or $y_A(s_2) > -.36$ then brings higher utility to A while $y_B(s_1) < .44$ or $y_B(s_2) > .44$ brings higher utility to B. Whether other distributional assumptions can produce equilibria which do not increase in s remains an open question.

3 goes further still. Welfare is given by the following integral,

$$W = \int_Z \sum_{S^2} g\left(s_A, s_B|z\right) \max\left\{u\left(x^*_{A,\infty}\left(s_A\right), z\right), u\left(x^*_{B,\infty}\left(s_B\right), z\right)\right\} f\left(z\right) dz \tag{5}$$

which can be computed numerically as $W_1 \approx -.073$, $W_2 \approx -.075$, and $W_3 \approx -.062$ for the three equilibria. While these welfare differences are not large, this makes clear that the polarization exhibited in equilibria 1 and 2 can be undesirable, and that the unusual behavior of equilibrium 3 can actually be socially optimal.

4.2 Three Signals

If $K = 3$ then $S = \{-1, 0, 1\}$ and $g\left(s|z\right) = \frac{1}{3}\left(1 + sz\right)$. There are many more possible platform configurations with three signal realizations than with two. It seems reasonable to focus attention on equilibria that are increasing in s, but even in that case, $y_B\left(s_1\right)$, $y_B\left(s_2\right)$, and $y_B\left(s_3\right)$ could all lie below $y_A\left(s_1\right)$, between $y_A\left(s_1\right)$ and $y_A\left(s_2\right)$, between $y_A\left(s_2\right)$ and $y_A\left(s_3\right)$, or above $y_A\left(s_3\right)$, or they could each lie in different cells of this partition. Despite so many possible configurations, only two can now be sustained in equilibrium, as explained below.

4.2.1 Equilibrium 1: polarization

If $y_A\left(s_1\right) < y_A\left(s_2\right) < y_A\left(s_3\right) < y_B\left(s_1\right) < y_B\left(s_2\right) < y_B\left(s_3\right)$ then the candidates are polarized, just as in equilibrium 1 of Section 4.1. An equilibrium of this type can be derived by extending (4) to include three realizations of s_A, and equating this to $y_B\left(s\right)$ simultaneously for each $s \in S$, while also solving analogous equations for candidate A. This yields a unique solution $y^*_{A,\infty} \approx (-.58, -.51, -.26)$, $y^*_{B,\infty} \approx (.26, .51, .58)$, which is symmetric around the origin. As in Section 4.1, this is notable in that candidate B is to the right of candidate A, even when A observes the most conservative signal and B observes the most liberal signal. In response to identical signals, the candidates adopt policy platforms that differ by between .84 and 1.02.

4.2.2 Equilibrium 2: extremist and centrist

If $y_A\left(s_1\right) < y_A\left(s_2\right) < y_B\left(s_1\right) < y_B\left(s_2\right) < y_B\left(s_3\right) < y_A\left(s_3\right)$ then candidate A adopts extreme policy positions, as in equilibrium 3 of Section 4.1, while B remains more centrist. An equilibrium of this type can be derived by equating an extended version of (3) to $y_B(s)$ for $s \in S$ and expressions analogous to solve $\lim_{n\to\infty} E(z|A, s; x_A < y_B(s_1) < y_B(s_2) < y_B(s_3)) = y_A(s)$ for $s \in \{s_1, s_2\}$ and $\lim_{n\to\infty} E(z|A, s; y_B(s_1) < y_B$ $y_A(s)$ for $s = s_3$, using expressions analogous to extensions of (2) and (4). This yields a unique solution $y^*_{A,\infty} \approx (-.67, -.63, .74)$, $y^*_{B,\infty} \approx (-.04, .14, .28)$.

As in equilibrium 2 of Section 4.1, adopting low positions for $y_A(s_1)$ and $y_A(s_2)$ is justified by the fact that, if A wins from this position, it will be because z is closer to these liberal positions than to the centrist positions adopted by candidate B. Similarly, the high position for $y_A(s_3)$ is justified because A will only win from that position if z is closer to this conservative position than to the centrist positions that B adopts. Meanwhile, B's centrist positions are justified by the expectation that they have beaten either one of candidate A's highly liberal positions or candidate A's highly conservative position.

4.2.3 No other equilibria

As in Section 4.1, an addition equilibrium with $y^*_{A,\infty} \approx (-.74, .63, .67)$ and $y^*_{B,\infty} \approx (.28, .14, -.04)$ mirrors equilibrium 2, and there are additional equilibria matching equilibria 1 and 2 but with candidates A and B trading roles. Other than these, however, no other equilibria exist: first-order conditions for equilibria with $y_A(s_1) < y_A(s_2) < y_B(s_1) < y_B(s_2) < y_A(s_3) < y_B(s_3)$ (or, symmetrically, $y_A(s_1) < y_B(s_1) < y_A(s_2) < y_A(s_3) < y_B(s_2) < y_B(s_3)$), $y_A(s_1) < y_B(s_1) < y_A(s_2) < y_B(s_2) < y_A(s_3) < y_B(s_3)$, or $y_A(s_1) < y_B(s_1) < y_B(s_2) < y_A(s_2) < y_A(s_3) < y_B(s_3)$ yield no solutions. $y_A \approx (-.60, -.54, .06)$ and $y_B \approx (-.06, .54, .60)$ satisfy first-order conditions but maximize utility only locally, not globally: $y_A(s_3) < -.06$ improves A's utility and $y_B(s_1) > .06$ improves B's. Similarly, $y_A \approx (-.71, -.24, .70)$ and $y_B \approx (-.38, .08, .25)$ (or, symmetrically, $y_A \approx (-.70, .24, .71)$ and $y_B \approx (-.25, -.08, .38)$) satisfy first-order conditions, but $y_A(s_2) < -.25$ improves A's utility and $y_B(s_1) > -.24$ improves B's.

4.2.4 Welfare

In Section 4.1, the centrist and extremist equilibrium generated higher welfare than the polarized equilibrium. Here, the reverse is true: evaluating (5) numerically yields $W_1 \approx -.074$ and $W_2 \approx -.085$. The key difference from before may be that, with only two signal realizations, equilibrium 1 of Section 4.1 runs the risk that a moderate realization of z will generate a negative signal for candidate A and a positive signal for candidate B, so that voters are offered only two competing extremes. With three signals, the choice between $y_A(s_1)$ and $y_B(s_3)$ is less likely; more probably, voters will always have at least one moderate option available when z is moderate (but also still have an extreme option when z is extreme).

4.3 Four Signals

If $K = 4$ then $S = \left\{-1, -\frac{1}{3}, \frac{1}{3}, 1\right\}$ and $g(s|z) = \frac{1}{4}(1 + sz)$. A comprehensive analysis of the many possible platform configurations is beyond the scope of this paper, but the following identifies three equilibrium possibilities, analogous to the equilibria above.

4.3.1 Equilibrium 1: polarization

As in Sections 4.1 and 4.2, there is an equilibrium in which candidates are polarized: extending (4) to include four realizations of s_A and equating this to $y_B(s)$ for each $s \in S$, along with analogous first-order conditions for candidate A, yields a unique solution with $y^*_{A,\infty} \approx (-.58, -.54, -.47, -.26)$ and $y^*_{B,\infty} \approx (.26, .47, .54, .58)$. As in previous sections, this equilibrium entails A taking a position left of center and B taking a position right of center. In response to the same signals, the candidates adopt platforms that differ by between .84 and 1.01. Both platform strategies increase in s, but even when $s_A = 1$ and $s_B = -1$, B is to the right of A.

4.3.2 Equilibrium 2: partial polarization

As in Section 4.1 (but unlike Section 4.2), a partially polarized equilibrium exists with $y_A(s_1) < y_B(s_1) < y_A(s_2) < y_B(s_2) < y_A(s_3) < y_B(s_3) < y_A(s_4) < y_B(s_4)$, which can be derived by extending (2) through (4) and analogous expressions for candidate A to allow four signal realizations, and equating these to $y_B(s)$ and $y_A(s)$. This again yields a unique solution, $y^*_{A,\infty} \approx (-.64, -.36, .10, .24)$ and $y^*_{B,\infty} \approx (-.24, -.10, .36, .65)$. Polarization is again only partial, in that either $y_A(s_A)$ or $y_B(s_B)$ may be larger. Still, polarization is substantial: in response to the same signal realization, candidates adopt platforms that differ by .26 to .40.

4.3.3 Equilibrium 3: centrist and extremist

As in Sections 4.1 and 4.2, there is an equilibrium in which candidate A adopts extreme platforms while B stays closer to the center. Once again, this can be derived by extending (2) through (4) and analogous expressions for candidate A to allow four signal realizations, and equating these to $y_B(s)$ and $y_A(s)$. The unique solution to these first-order conditions is $y^*_{A,\infty} \approx (-.40, -.30, .30, 40)$ and $y^*_{B,\infty} \approx (-.09, -.03, .03, .09)$. The logic underlying this equilibrium is the same as before: whether A wins from a liberal or a conservative position, it is because z is closer to this position than to the center, but if B wins it might be because z is low and s_A is high or because z is high and s_A is low, and only the center provides a safe hedge against joining A on the wrong side of the policy interval.

4.3.4 Welfare

The welfare associated with the three equilibria above can once again be computed numerically using (5). This yields $W_1 \approx -.076$, $W_2 \approx -.148$, and $W_3 \approx -.122$. Again, the ranking of these equilibria can be understood in terms of their tendency to offer voters a policy near z. Because y_A and y_B overlap in equilibrium 2, the electorate runs the risk of both candidates adopting policies on the opposite end of the policy spectrum from z. Equilibrium reduces this problem by assuing that at least one candidate's policy is centrist. Equilibrium 1 is socially optimal, as it guarantees voters a policy choice on both sides of the policy interval. Moreover, these both tend to be low when z is low and high when z is high.

4.4 Multiple Signals

As K increases, numerical analysis becomes more computationally cumbersome. However, the symmetry of equilibrium 1 in each of the sections above allows this equilibrium to be extended to $y^*_{B,\infty} \approx (.26, .42, .51, .55, .58)$ for $K = 5$ and $y^*_{B,\infty} \approx (.26, .41, .48, .53, .56, .58)$ for $K = 6$, with symmetric platforms for A. Evidently, polarization by candidates with the most extreme signals is not highly dependent on the number of intermediate signal realizations.

As K grows large, $g(s|z)$ approaches the density $g(s|z) = \frac{1}{2}(1 + sz)$, defined on $S = [-1, 1]$. Equilibrium must then satisfy a continuum of first-order conditions, and computating this is beyond the scope of this paper. To gain some insight, however, consider a strategy $y_A(s) = \frac{1}{2}(s+1)$ that is simply linear in s, so that $y_A(s)$ is uniform on $[-1, 0]$. A best response for candidate B (given private signal s_B) must satisfy $E(z|B, s_B) = y_B$, but it can be shown numerically that this is satisfied for $y_B \approx 0.06$, and that $E(z|B, s_B) - y_B$ is positive for all smaller values of y_B, including all negative values. This does not characterize an equilibrium, but does establish that candidates' pivotal calculus can polarize their best response behavior, just as in the models with smaller K.

5 Conclusion

It has long been recognized that voters should strategically restrict attention to pivotal events, no matter how unlikely, and that this pivotal calculus can dramatically alter voter behavior, especially in common-interest settings. Recent work has brought attention to a pivotal calculus for candidates, but only in a setting where candidates lack private information of their own. This paper has showed that adding candidate signals preserves this strategic incentive and, in fact, highlights how dramatically this calculus alters candidates' behavior, just as it alters voters'.

The first impact of pivotal considerations is to make a candidate's utility non-monotonic in her policy choice: when she is adopts a policy left of her opponent's, she infers from the event of winning that z was low; when she adopts a policy right of her opponent's, she infers that z was high. One consequence of this is that identical candidates never behave identically in equilibrium. In some cases, candidates may behave quite differently from each other: for example, one might react strongly to her private signal, adopting a policy at one of the two extreme ends of the policy space, while the other barely reacts at all to her signal, always staying close to the political center.

A second consequence of the pivotal calculus is that multiple equilibria can arise. If culture or history somehow designates party A as being liberal and party B as being conservative, for example, the two candidates have incentive to follow those designations, even when candidate A privately believes a conservative policy to be optimal and candidate B privately believes a liberal policy to be optimal. Of course, the reverse can occur in equilibrium, as well. Either way, polarization emerges fairly robustly as a way of sorting the informational content inherent in winning elections. Indeed, candidates may be so polarized that they react to the same signal with wildly different policies.[9] Such polarization can be detrimental to voter welfare, although it can also be optimal, depending on model specifics. In general, the key determinant of equilibrium welfare is how reliably the candidates offer voters a choice that is close to the truly optimal policy.

This analysis assumes that voters' signals are reliable enough to satisfy the requirements of Condorcet's (1785) jury theorem. An important direction for future work is to consider electorates who have imperfect information, even collectively. Presumably, the pivotal inference would be less likely to overwhelm candidates' private judgments in that case. Of course, the analysis also assumes that voters and candidates share fundamentally common interests. If candidates do not share voters' preferences, they may react differently to voters' information, and this may in turn alter voters' incentives.

Empirically, it is not clear that candidates infer important information from voters any more than it is clear that voters infer information from one another.[10] One possibility is that candidates do make use of this information, perhaps subconsciously: it would be difficult to distinguish empirically whether a candidate favors a policy because of personal opinions alone, or because of personal opinion combined with an assessment of voter support. Another possibility is that candidates make no such inference, either because they lack the strategic sophistication or because inference that is useful in the specialized model above is less useful in richer models that more closely depict reality. Regardless, a stylized model is useful both for clarifying which model elements might change the main results, and as a benchmark to which more elaborate models can be compared.

With pure common interest and an asymptotically infallible electorate, this model is admittedly idealized, but the main result seems likely to be robust even in environments with richer preference heterogeneity or other realistic feature: to the extent that voters' and candidates' policy preferences rely on the same unknown facts about the policy environment, the general principle should still apply that the pivotal event of winning the election should at least partially reveal voters' information to candidates, and candidates who utilize this information should have higher utility than

[9] In McMurray (2017, 2020) I emphasize the possibility of a binary state variable. In the present setting, this would polarize candidates even more dramatically, as the one on the left is sure to win if and only if $z = -1$ while the one on the right wins when $z = 1$.

[10] Laboratory experiments on voting behavior demonstrate a striking inability to infer information from pivotal events (Esponda and Vespa, 2014).

candidates who do not.

A Appendix

Proof of Proposition 1. Using (1) and conditional on $\tilde{s}_i = \tilde{s}$, $y_A(s_A) = x_A$, and $y_B(s_B) = x_B$, the difference in expected utility between these two vote choices can therefore be written as follows.

$$\begin{aligned} \Delta(\tilde{s}) &= E\{[u(x_B, z) - u(x_A, z)] \Pr(P|z) | \tilde{s}, S_A(x_A), S_B(x_B)\} \\ &= 2(x_B - x_A) \Pr(P|\tilde{s}, S_A(x_A), S_B(x_B)) \{E[z|P, \tilde{s}, S_A(x_A), S_B(x_B)] - \bar{x}\} \end{aligned} \quad (6)$$

This expression is positive for any voter whose private signal is such that $E[z|P, \tilde{s}, S_A(x_A), S_B(x_B)]$ is closer to x_B than to x_A. Condition 3 guarantees that this expectation is monotonic in $\tilde{s}$, implying the existence of a best response threshold $t^{br} \in \tilde{S}$ such that voters with signals above and below t^{br} vote for the more conservative candidate and the more liberal candidate, respectively. In other words, best response voting is ideological, and $E\left[z|P, \tilde{s}_i = t^{br}, S_A(x_A), S_B(x_B)\right] = \frac{x_A + x_B}{2}$, as claimed.

When $x_A < x_B$ and other voters follow an ideological strategy v_t, an increase in t makes voters more likely to vote A. As I show in McMurray (2017a), this increases the distribution of $\Pr(P|z)$ and, by Bayes' rule, the distribution of $f(z|P)$, in the sense of first-order stochastic dominance, so that $E[z|P, \tilde{s}, S_A(x_A), S_B(x_B)]$ and therefore $\Delta(\tilde{s})$ increase, making a voter more willing to vote B in response, so that t^{br} falls. In other words, $t^{br}(t)$ decreases in t, implying a unique fixed point $t^* \in \tilde{S}$ that characterizes the equilibrium voting response to x_A and x_B. From (6) it is clear that the location of the indifferent voter depends only on the midpoint $\bar{x}$ of the two platforms. As $\Delta(\tilde{s})$ decreases in $\bar{x}$, the voter with $\tilde{s}_i = \tilde{s}$ prefers to vote A, implying that $t^{br}(t) > t$. As t^{br} decreases in t, the new fixed point increases to $t^* > t$. Symmetric reasoning applies if $x_A > x_B$.

A strategy $v \in V$ is alternatively characterized by the indicator function $1_{v(s)=j}$ that equals one if $v(s) = j$ and zero otherwise. The set of such functions is closed and bounded in the metric topology and is uniformly integrable, so it is compact in the weak topology (Duggan, 2013). The probability $\phi(j|z) = \int_S 1_{v(s)=j} g(s|z)$ of voting for candidate $j \in \{A, B\}$ in state $z \in Z$ is weakly continuous in this indicator function, and expected utility is continuous in $\phi(j|z)$, so by the Weierstrass theorem, there exists some $1_{v(s)=j}$ or, equivalently, some $v \in V$, that produces maximal expected utility. By the logic of McLennan (1998), this social optimum also constitutes a BNE in the voting subgame. If $x_A \neq x_B$ then the unique BNE v^* is also uniquely optimal in V. ■

Proof of Lemma 2. If candidate A plays strategy $y_A \in Y$ and candidate B responds with $y_B(s_B) = x_B \in X$ then candidate B's expected utility can be written as follows.

$$\begin{aligned}
&\lim_{n\to\infty} E_{s_A,w,z}\left[u(x_w,z)|s_B\right] \\
=\;& \sum\nolimits_{s_A} \int_z \left[\begin{array}{c} \Pr(A|y_A(s_A),z;x_B)\,u(y_A(s_A),z) \\ +\Pr(B|y_A(s_A),z;x_B)\,u(x_B,z) \end{array}\right] f(z|s_A,s_B)\,g(s_A|s_B) \\
=\;& \sum_{s_A:x_A(s_A)<x_B} \int_z \left[\begin{array}{c} \Pr(A|y_A(s_A),z;x_B)\,u(y_A(s_A),z) \\ +\Pr(B|y_A(s_A),z;x_B)\,u(x_B,z) \end{array}\right] f(z|s_A,s_B)\,g(s_A|s_B) \\
&+ \sum_{s_A:x_A(s_A)>x_B} \int_z \left[\begin{array}{c} \Pr(A|y_A(s_A),z;x_B)\,u(y_A(s_A),z) \\ +\Pr(B|y_A(s_A),z;x_B)\,u(x_B,z) \end{array}\right] f(z|s_A,s_B)\,g(s_A|s_B) \quad (7)
\end{aligned}$$

As n grows large, the jury thoerem implies that $\lim_{n\to\infty} \Pr(B|y_A(s_A),z;x_B) = \begin{cases} 0 \text{ if } |z-y_A(s_A)|<|z-x_B| \\ 1 \text{ if } |z-y_A(s_A)|>|z-x_B| \end{cases}$, so this expression converges uniformly to the following,

$$\begin{aligned}
&\lim_{n\to\infty} E_{s_A,w,z}\left[u(x_w,z)|s_B\right] \\
=\;& \sum_{s_A:x_A(s_A)<x_B} \int_z \left[\begin{array}{c} 1\left(z<\frac{y_A(s_A)+x_B}{2}\right)u(y_A(s_A),z) \\ +1\left(z>\frac{y_A(s_A)+x_B}{2}\right)u(x_B,z) \end{array}\right] f(z|s_A,s_B)\,g(s_A|s_B) \\
&+ \sum_{s_A:x_A(s_A)>x_B} \int_z \left[\begin{array}{c} 1\left(z>\frac{y_A(s_A)+x_B}{2}\right)u(y_A(s_A),z) \\ +1\left(z<\frac{y_A(s_A)+x_B}{2}\right)u(x_B,z) \end{array}\right] f(z|s_A,s_B)\,g(s_A|s_B) \\
=\;& \sum_{s_A:y_A(s_A)<x_B} \int_{-1}^{\frac{y_A(s_A)+x_B}{2}} u(y_A(s_A),z)\,f(z|s_A,s_B)\,g(s_A|s_B) \\
&+ \sum_{s_A:y_A(s_A)<x_B} \int_{\frac{y_A(s_A)+x_B}{2}}^{1} u(x_B,z)\,f(z|s_A,s_B)\,g(s_A|s_B) \\
&+ \sum_{s_A:y_A(s_A)>x_B} \int_{\frac{y_A(s_A)+x_B}{2}}^{1} u(y_A(s_A),z)\,f(z|s_A,s_B)\,g(s_A|s_B) \\
&+ \sum_{s_A:y_A(s_A)>x_B} \int_{-1}^{\frac{y_A(s_A)+x_B}{2}} u(x_B,z)\,f(z|s_A,s_B)\,g(s_A|s_B)
\end{aligned}$$

where $1(\cdot)$ is an indicator function that equals one if the inequality in parentheses holds and equals zero otherwise.

As long as x_B differs from $y_A(s)$ for every $s \in S$, differentiating this function yields the following.

$$\frac{d}{dx_B}\lim_{n\to\infty} E_{s_A,w,z}\left[u\left(x_w,z\right)|s_B\right]$$

$$= \left\{ \begin{array}{c} \frac{1}{2}\sum_{s_A:y_A(s_A)<x_B} u\left(y_A\left(s_A\right), \frac{y_A(s_A)+x_B}{2}\right) f\left(\frac{y_A(s_A)+x_B}{2}|s_A,s_B\right) g\left(s_A|s_B\right) \\ -\frac{1}{2}\sum_{s_A:y_A(s_A)<x_B} u\left(x_B, \frac{y_A(s_A)+x_B}{2}\right) f\left(\frac{y_A(s_A)+x_B}{2}|s_A,s_B\right) g\left(s_A|s_B\right) \\ +2\sum_{s_A:y_A(s_A)<x_B} \int_{\frac{y_A(s_A)+x_B}{2}}^{1} \left(z-x_B\right) f\left(z|s_A,s_B\right) g\left(s_A|s_B\right) \\ -\frac{1}{2}\sum_{s_A:y_A(s_A)>x_B} u\left(y_A\left(s_A\right), \frac{y_A(s_A)+x_B}{2}\right) f\left(\frac{y_A(s_A)+x_B}{2}|s_A,s_B\right) g\left(s_A|s_B\right) \\ +\frac{1}{2}\sum_{s_A:y_A(s_A)>x_B} u\left(x_B, \frac{y_A(s_A)+x_B}{2}\right) f\left(\frac{y_A(s_A)+x_B}{2}|s_A,s_B\right) g\left(s_A|s_B\right) \\ +2\sum_{s_A:y_A(s_A)>x_B} \int_{-1}^{\frac{y_A(s_A)+x_B}{2}} \left(z-x_B\right) f\left(z|s_A,s_B\right) g\left(s_A|s_B\right) \end{array} \right\}$$

Since $u\left(y_A\left(s_A\right), \frac{y_A(s_A)+x_B}{2}\right) = u\left(x_B, \frac{y_A(s_A)+x_B}{2}\right)$, this reduces as follows,

$$\begin{aligned} & \frac{d}{dx_B}\lim_{n\to\infty} E_{s_A,w,z}\left[u\left(x_w,z\right)|s_B\right] \\ = & \left\{ \begin{array}{c} 2\sum_{s_A:y_A(s_A)<x_B} \int_{\frac{y_A(s_A)+x_B}{2}}^{1} \left(z-x_B\right) f\left(z|s_A,s_B\right) g\left(s_A|s_B\right) \\ +2\sum_{s_A:y_A(s_A)>x_B} \int_{-1}^{\frac{y_A(s_A)+x_B}{2}} \left(z-x_B\right) f\left(z|s_A,s_B\right) g\left(s_A|s_B\right) \end{array} \right\} & (8) \\ = & \Pr\left(B_R|s_B\right)\left[E\left(z|B_R,s_B\right)-x_B\right] + \Pr\left(B_L|s_B\right)\left[E\left(z|B_L,s_B\right)-x_B\right] \\ = & \Pr\left(B|s_B\right)\left[E\left(z|B,s_B\right)-x_B\right] & (9) \end{aligned}$$

where $B_L = \left(x_B < y_A\left(s_A\right)\right) \cap \left(|z-x_B| < |z-y_A\left(s_A\right)|\right)$ and $B_R = \left(y_A\left(s_A\right) < x_B\right) \cap \left(|z-x_B| < |z-y_A\left(s_A\right)|\right)$ denote the events of candidate B winning the election from a position that is to the left and to the right of candidate A, respectively, and $B = B_L \cup B_R$. Any sequence of equilibria must satisfy $\frac{d}{dx_B}E_{s_A,w,z}\left[u\left(x_w,z\right)|s_B\right] = 0$ for all n, so (9) must equal zero. Clearly, this holds if and only if $\lim_{n\to\infty}\left[y^*_{B,n}\left(s_B\right) - E\left(z|B,s_B\right)\right] = 0$. An analogous result holds for candidate A.

Differentiating expected utility a second time yields the following.

$$\begin{aligned} & \frac{d^2}{dx_B^2}\lim_{n\to\infty} E_{s_A,w,z}\left[u\left(x_w,z\right)|s_B\right] & (10) \\ = & \left[E\left(z|B,s_B\right)-x_B\right]\frac{d}{dx_B}\Pr\left(B|s_B\right) + \Pr\left(B|s_B\right)\left[\frac{d}{dx_B}E\left(z|B,s_B\right)-1\right] & (11) \end{aligned}$$

The second difference in brackets is always negative, because $f\left(z|s_A,s_B\right)$ is log-concave for all $\left(s_A,s_B\right) \in S^2$, so as Bagnoli and Bergstrom (2005) show, $\frac{d}{d\bar{x}}E\left(z|z<\bar{x},s_B\right) < 1$ and $\frac{d}{d\bar{x}}E\left(z|z>\bar{x},s_B\right) < 1$. If s_A is such that $x_A\left(s_A\right) < x_B$ then $\frac{d}{dx_B}E\left(z|B,s_A,s_B\right) = E\left(z|z > \frac{x_A(s_A)+x_B}{2}, s_A, s_B\right) = \frac{1}{2}\frac{d}{d\bar{x}}E\left(z|z>\bar{x},s_B\right) < \frac{1}{2}$, and similarly, $x_A\left(s_A\right) > x_B$ implies that $\frac{d}{dx_B}E\left(z|B,s_A,s_B\right) = E\left(z|z < \frac{x_A(s_A)+x_B}{2}, s_A, s_B\right) = \frac{1}{2}\frac{d}{d\bar{x}}E\left(z|z<\bar{x},s_B\right) < \frac{1}{2} < 1$. Together, these imply that $\frac{d}{dx_B}E(z|B,s_B)=\sum_S g(s_A|B,s_B)\frac{d}{dx_B}E(z|B,s_A,s_B)<\frac{1}{2}$. When (9) equals zero, the first difference in brackets is zero, as well, implying that (10) is negative and expected utility is locally concave whenever the first-order equilibrium condition is satisfied.

So far, this analysis has assumed that $x_B \neq y_A(s)$ for any $s \in S$. If $x_B = y_A(\bar{s})$ for some $\bar{s} \in S$ then expected utility is discontinuous in x_B. Approaching $y_A(\bar{s})$ from the left, (9) can be written as follows.

$$\begin{aligned}
& \lim_{x_B \to_- y_A(\bar{s})} \frac{d}{dx_B} \lim_{n\to\infty} E_{s_A,w,z}\left[u(x_w,z)|s_B\right] \\
= & \sum_{s_A: y_A(s_A)<y_A(\bar{s})} \int_{\frac{y_A(s_A)+y_A(\bar{s})}{2}}^{1} (z-y_A(\bar{s}))\, f(z|s_A,s_B)\, g(s_A|s_B) \\
& + \sum_{s_A: y_A(s_A)\geq y_A(\bar{s})} \int_{-1}^{\frac{y_A(s_A)+y_A(\bar{s})}{2}} (z-y_A(\bar{s}))\, f(z|s_A,s_B)\, g(s_A|s_B)
\end{aligned}$$

Approaching from the right yields an identical expression, except with strict and weak inequalities reversed. The difference is given as follows,

$$\begin{aligned}
& \lim_{x_B \to_+ y_A(\bar{s})} \frac{d}{dx_B} \lim_{n\to\infty} E_{s_A,w,z}\left[u(x_w,z)|s_B\right] - \lim_{x_B \to_- y_A(\bar{s})} \frac{d}{dx_B} \lim_{n\to\infty} E_{s_A,w,z}\left[u(x_w,z)|s_B\right] \\
= & \int_{y_A(\bar{s})}^{1} (z-y_A(\bar{s}))\, f(z|\bar{s},s_B)\, g\left(y_A^{-1}(x_B)|s_B\right) - \int_{-1}^{y_A(\bar{s})} (z-y_A(\bar{s}))\, f(z|\bar{s},s_B)\, g\left(y_A^{-1}(x_B)|s_B\right) \\
= & \Pr(z>y_A(\bar{s}))\left[E(z|z>y_A(\bar{s}))-y_A(\bar{s})\right]+\Pr(z<y_A(\bar{s}))\left[y_A(\bar{s})-E(z|z<y_A(\bar{s}))\right]
\end{aligned}$$

which is clearly positive. Thus, marginal utility is higher to the right of $y_A(\bar{s})$ than to the left, implying that utility is not maximized locally at $x_B = y_A(\bar{s})$.

Together the arguments above guarantee parts 1 and 2 of the proposition. Part 3 follows as a corollary, because events B_L and B_R are the same for a candidate with signal s' or s'', as long as there is no $s \in S$ such that $x_A(s) \in [x_B(s'), x_B(s'')]$, but the monotone likelihood ratio property of $g(s|z)$ implies that $E(z|B_L,s_B)$ and $E(z|B_R,s_B)$ both increase in s_B, so from (9), $E(z|B,s')=0$ implies that $E(z|B,s'')>0$. ■

References

[1] Austen-Smith, David and Jeffrey S. Banks. 1996. "Information Aggregation, Rationality, and the Condorcet Jury Theorem." *The American Political Science Review*, 90(1): 34-45.

[2] Bagnoli, Mark, and Ted Bergstrom. 2005. "Log-Concave Probability and Its Applications." *Economic Theory*, 26(2): 445-469.

[3] Balder, Erik J. 1988. "Generalized Equilibrium Results for Games with Incomplete Information." *Mathematics of Operations Research, 13(2): 265-276.*

[4] Condorcet, Marquis de. 1785. *Essay on the Application of Analysis to the Probability of Majority Decisions*. Paris: De l'imprimerie royale. Trans. Iain McLean and Fiona Hewitt. 1994.

[5] Duggan, John. 2013. "A Survey of Equilibrium Analysis in Spatial Models of Elections." Working paper, University of Rochester.

[6] Esponda, Ignacio and Emanuel Vespa. 2014. "Hypothetical Thinking and Information Extraction in the Laboratory." *American Economic Journal: Microeconomics*, 6(4): 180-202.

[7] Feddersen, Timothy J. and Wolfgang Pesendorfer. 1996. "The Swing Voter's Curse." *The American Economic Review*, 86(3): 408-424.

[8] Feddersen, Timothy J. and Wolfgang Pesendorfer. 1998. "Convicting the Innocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting." *The American Political Science Review*, 92(1): 23-35.

[9] McLennan, Andrew. 1998. "Consequences of the Condorcet Jury theorem for Beneficial Information Aggregation by Rational Agents." *American Political Science Review*, 92(2): 413-418.

[10] McMurray, Joseph C. 2017. "Ideology as Opinion: A Spatial Model of Common-value Elections." *American Economic Journal: Microeconomics*, 9(4): 108-140.

[11] McMurray, Joseph C. 2020. "Polarization and Pandering in Common Interest Elections." Working paper, Brigham Young University.

[12] Myerson, Roger. 1998. "Population Uncertainty and Poisson Games." *International Journal of Game Theory*, 27: 375-392.

[13] Tajika, Tomoya. 2019. "Collective Mistake in a Tricky Question Under Strategic Voting." Working paper, Hokusei Gakuen University.